

Question 1

Consider the initial value problem

$$(1 + x^2)y' + 2xy = 3x^2 + 1, \quad y(0) = 2.$$

Tasks

1. Put the equation in standard linear form $y' + p(x)y = q(x)$ and find an integrating factor. Explain why multiplying by that factor produces a product derivative.
2. Find the general solution and then the solution of the IVP.
3. Verify the IVP solution in the original, unnormalized equation and state its largest open interval.
4. Identify a straight-line particular solution and determine whether every solution approaches that line as $x \rightarrow +\infty$ and as $x \rightarrow -\infty$. State precisely what approaches zero.

Question 1 – Solution

Strategy. Normalize first. Choose μ with $\mu' = p\mu$, so the product rule gives $(\mu y)' = \mu(y' + py)$.

Step 1: Integrating factor. Since $1 + x^2 > 0$,

$$y' + \frac{2x}{1+x^2}y = \frac{3x^2+1}{1+x^2}, \quad \mu = e^{\int 2x/(1+x^2) dx} = 1+x^2.$$

Multiplication restores the original left side as $[(1+x^2)y]'$.

Step 2: Integrate and select the constant. We obtain

$$(1+x^2)y = \int (3x^2+1) dx = x^3 + x + C,$$

so

$$y = x + \frac{C}{1+x^2}.$$

At $x = 0$, $C = 2$. The IVP solution is therefore $y = x + 2/(1+x^2)$.

Step 3: Direct check and interval. Its derivative is $y' = 1 - 4x/(1+x^2)^2$, and

$$\begin{aligned}(1+x^2)y' + 2xy &= 1+x^2 - \frac{4x}{1+x^2} + 2x^2 + \frac{4x}{1+x^2} \\ &= 3x^2 + 1.\end{aligned}$$

Also $y(0) = 2$. No denominator vanishes for real x , so the largest open interval is $\mathbb{R}$.

Step 4: Approach to a line. Setting $C = 0$ gives the particular solution $y_p = x$. For every fixed real C ,

$$y(x) - x = \frac{C}{1+x^2} \longrightarrow 0 \quad \text{as } x \rightarrow \pm\infty.$$

It is the vertical difference from the line that tends to zero; the solution itself does not tend to a finite constant.

Question 2

On the domain $x > 0$, consider

$$xy' + 2y = x^2 \ln x, \quad y(1) = 0.$$

Tasks

1. Solve the IVP using an integrating factor, showing the integration-by-parts step needed for the forcing term.
2. State the largest open interval containing 1 on which the solution solves this equation, and verify the initial value and differential equation.
3. Rewrite the solution using a definite integral based at 1. Use the sign of that integral to decide whether the solution can be negative.
4. Locate the global minimum on $x > 0$ and sketch the solution near $x = 1$ in your solution. Explain why positive forcing only for $x > 1$ does not make the solution negative for $0 < x < 1$.

Question 2 – Solution

Strategy. Use x^2 as integrating factor, then retain a definite-integral form to analyze signs on both sides of the initial point.

Step 1: Solve. Dividing by x gives $y' + 2y/x = x \ln x$. With $\mu = x^2$,

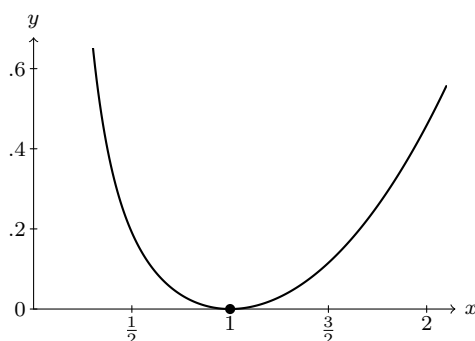
$$(x^2 y)' = x^3 \ln x.$$

Integration by parts, $u = \ln x$, $dv = x^3 dx$, yields

$$\int x^3 \ln x \, dx = \frac{x^4}{4} \ln x - \frac{x^4}{16} + C.$$

The initial value forces $C = 1/16$, hence

$$y = \frac{x^2}{4} \ln x - \frac{x^2}{16} + \frac{1}{16x^2}, \quad x > 0.$$



The solution is positive except at its global minimum $(1, 0)$.

Step 2: Verify. Substitution at 1 gives zero. Differentiating $x^2 y$ gives $x^3 \ln x$, or $x^2 y' + 2xy = x^3 \ln x$; division by $x > 0$ recovers the original equation. The logarithm and coefficient division restrict the interval to $(0, \infty)$.

Step 3: Sign and minimum. Equivalently,

$$x^2 y(x) = \int_1^x t^3 \ln t \, dt.$$

For $x > 1$ the integral is positive. For $0 < x < 1$, the integrand is negative between x and 1, but the reversed limits make the integral positive. Since $x^2 > 0$, $y(x) > 0$ for $x \neq 1$. Thus $\min_{x>0} y = 0$ only at $x = 1$. The sign of the forcing alone does not determine the sign of the solution.

Question 3

A student tries to solve

$$y' - 2y = xe^{2x}, \quad y(0) = 0,$$

and writes

$$\mu = e^{2x}, \quad (e^{2x}y)' = xe^{4x}.$$

Tasks

1. Identify the precise error by expanding the student's claimed product derivative.
2. Find a correct integrating factor and solve the IVP.
3. Verify the solution by direct substitution in $y' - 2y = xe^{2x}$.
4. Replace the initial condition by $y(0) = a$, with a any real number. Find the resulting family and prove directly that two different choices of a cannot produce intersecting solution curves.

Question 3 – Solution

Strategy. The integrating factor must match the signed coefficient of y . Check its product derivative before integrating.

Step 1: Diagnose the sign error. The student's derivative is

$$(e^{2x}y)' = e^{2x}(y' + 2y),$$

whereas multiplying the original equation by e^{2x} gives $e^{2x}(y' - 2y)$. These expressions differ in the sign of the y term. The required coefficient is $p = -2$, so $\mu = e^{-2x}$.

Step 2: Correct integration. Multiplication gives

$$(e^{-2x}y)' = x, \quad e^{-2x}y = \frac{x^2}{2} + C.$$

Since $y(0) = 0$, $C = 0$, and

$$y = \frac{x^2}{2}e^{2x}, \quad x \in \mathbb{R}.$$

Step 3: Verify. Differentiating explicitly,

$$y' = xe^{2x} + x^2e^{2x}, \quad y' - 2y = xe^{2x} + x^2e^{2x} - x^2e^{2x} = xe^{2x}.$$

The initial value is also zero.

Step 4: Vary the initial datum. The general initial value sets $C = a$, giving

$$y_a(x) = \left(a + \frac{x^2}{2}\right)e^{2x}.$$

For distinct a, b ,

$$y_a(x) - y_b(x) = (a - b)e^{2x} \neq 0$$

at every real x . Hence their graphs never intersect. The nonintersection follows from the explicit homogeneous difference, not merely from a sketch.

Question 4

Consider the linear equation

$$y' - y = e^{-x}, \quad x \geq 0.$$

Instead of prescribing an initial value, require that the solution remain bounded on the entire half-line $[0, \infty)$.

Tasks

1. Find the general solution using an integrating factor.
2. Determine the unique bounded solution and the initial value it must have.
3. If that initial value is changed by a nonzero amount ε , find the difference from the bounded solution and describe its behavior as $x \rightarrow \infty$.
4. Sketch the bounded solution and the solutions for $\varepsilon = \pm 1/20$ in your solution. Explain why a forcing term tending to zero does not force every solution to tend to zero.

Question 4 – Solution

Strategy. Separate the forced response from the homogeneous term. A condition at infinity can select the constant just as an initial value can.

Step 1: General solution. With $\mu = e^{-x}$,

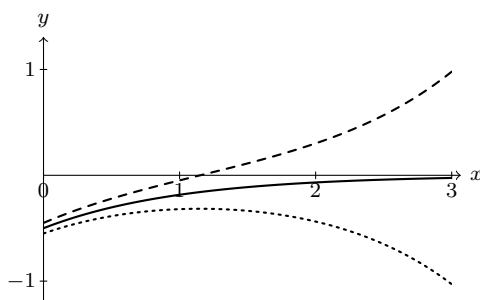
$$(e^{-x}y)' = e^{-2x}, \quad e^{-x}y = -\frac{1}{2}e^{-2x} + C.$$

Thus $y = Ce^x - \frac{1}{2}e^{-x}$; each formula is defined on $\mathbb{R}$.

Step 2: Impose boundedness. As $x \rightarrow \infty$, the second term tends to zero. If $C \neq 0$, the term Ce^x is unbounded and cannot be canceled by that decaying term. Therefore

$$y_b = -\frac{1}{2}e^{-x}, \quad y_b(0) = -\frac{1}{2}.$$

This solution is bounded and tends to zero, proving both existence and uniqueness under the stated boundedness condition.



Solid: bounded solution. Dashed: $\varepsilon = 1/20$. Dotted: $\varepsilon = -1/20$.

Step 3: Perturb the selected initial value. Since $y(0) = C - 1/2$, changing it to $-1/2 + \varepsilon$ gives $C = \varepsilon$. Consequently

$$y_\varepsilon - y_b = \varepsilon e^x.$$

For $\varepsilon > 0$ the perturbed solution tends to $+\infty$; for $\varepsilon < 0$ it tends to $-\infty$. Direct differentiation verifies $y' - y = e^{-x}$ for every C . The vanishing forcing does not remove a growing homogeneous component.

Question 5

Two functions are known to solve the same first-order equation in standard linear form,

$$y' + p(x)y = q(x), \quad x \in \mathbb{R},$$

where p, q are continuous. The functions are

$$y_1 = x + e^{-x}, \quad y_2 = x - e^{-x}.$$

Tasks

1. Recover $p(x)$ and $q(x)$ from these two solutions. Explain why they are determined at every real x .
2. Solve the recovered equation using an integrating factor to obtain all its solutions.
3. Determine all real constants α, β for which $\alpha y_1 + \beta y_2$ is also a solution of the same nonhomogeneous equation.
4. Explain why arbitrary sums of solutions need not remain solutions, although the difference of two solutions always satisfies the associated homogeneous equation.

Question 5 – Solution

Strategy. Subtract the two equations to eliminate the forcing. A nonzero solution difference reveals the coefficient p .

Step 1: Recover the coefficients. Let $d = y_1 - y_2 = 2e^{-x}$. Subtraction gives $d' + pd = 0$, so

$$p = -\frac{d'}{d} = 1,$$

valid for every real x because d never vanishes. Using $y_1' = 1 - e^{-x}$,

$$q = y_1' + y_1 = 1 - e^{-x} + x + e^{-x} = x + 1.$$

Thus $\boxed{y' + y = x + 1}$ is uniquely recovered within the stated standard linear form.

Step 2: All solutions. The integrating factor is e^x , giving

$$(e^x y)' = (x + 1)e^x = (xe^x)'.$$

Integration yields $\boxed{y = x + Ce^{-x}, \quad C \in \mathbb{R}}$. Substitution confirms $y' + y = x + 1$ for every constant.

Step 3: Which combinations work? Write $L[v] = v' + v$. Linearity gives

$$L[\alpha y_1 + \beta y_2] = (\alpha + \beta)(x + 1).$$

For this to equal $x + 1$ throughout an interval, we require

$$\boxed{\alpha + \beta = 1.}$$

These combinations are $x + (\alpha - \beta)e^{-x}$ and generate every solution.

Step 4: Nonhomogeneous versus homogeneous. The sum $y_1 + y_2 = 2x$ produces $L[2x] = 2x + 2$, not $x + 1$. The difference instead satisfies $L[y_1 - y_2] = q - q = 0$. Thus differences solve the homogeneous equation; combinations with coefficients summing to one preserve the original forcing.

Question 6

For each real initial value a , consider

$$y' + 2xy = 1, \quad y(0) = a.$$

A definite integral is an acceptable exact expression; no elementary antiderivative is required.

Tasks

1. Use an integrating factor to obtain an exact solution based at $x = 0$, and state its interval of definition.
2. Verify the formula by differentiation and the initial condition directly.
3. For solutions with initial values a and b , determine their difference, its greatest absolute value on $\mathbb{R}$, and whether their curves can cross when $a \neq b$.
4. Show that the solution with $a = 0$ is odd and positive for $x > 0$. Sketch the curves for $a = -1, 0, 1$ in your solution.

Question 6 – Solution

Strategy. Use a definite integral to incorporate the initial value exactly. The homogeneous difference makes comparison easier than evaluating the integral.

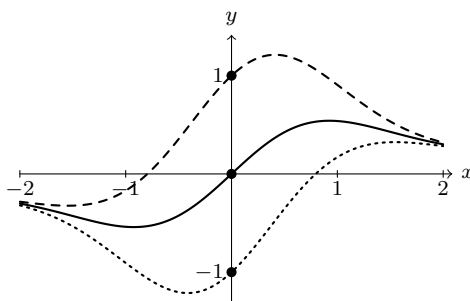
Step 1: Integrating factor and solution. Here

$$\mu = e^{\int 2x dx} = e^{x^2}, \quad (e^{x^2} y)' = e^{x^2}.$$

Integrating from 0 to x gives

$$y_a(x) = e^{-x^2} \left(a + \int_0^x e^{t^2} dt \right), \quad x \in \mathbb{R}.$$

The integral exists for every finite real x .



Solid: $a = 0$. Dashed: $a = 1$. Dotted: $a = -1$. The curves never cross.

Step 2: Direct verification. The product rule and Fundamental Theorem give $y'_a = -2xy_a + e^{-x^2}e^{x^2} = 1 - 2xy_a$. Also the integral is zero at 0, so $y_a(0) = a$.

Step 3: Compare initial values. The integral cancels in the difference:

$$y_a - y_b = (a - b)e^{-x^2}, \quad \max_{x \in \mathbb{R}} |y_a - y_b| = |a - b|.$$

For $a \neq b$, equality in the maximum occurs only at $x = 0$, and the difference never changes sign or vanishes at a finite point. The curves cannot cross.

Step 4: Symmetry of the zero-data solution. Since e^{t^2} is even, $\int_0^x e^{t^2} dt$ is odd. Multiplying by the even factor e^{-x^2} preserves oddness. For $x > 0$, both the integral and prefactor are positive, so $y_0(x) > 0$.

Question 7

Consider the periodically forced equation

$$y' + 2y = 3 \cos x + \sin x,$$

with initial condition $y(0) = 0$.

Tasks

1. Solve the IVP using an integrating factor. Show how you evaluate the integral of the forcing times that factor.
2. Find the unique 2π -periodic solution of the equation, and verify both its periodicity and its differential equation.
3. Find the earliest $X \geq 0$ such that the IVP solution differs from the periodic solution by at most $1/100$ for every $x \geq X$.
4. Sketch the IVP solution and the periodic solution on $[0, 2\pi]$ in your solution. Explain which part of the general solution is the transient and why the prescribed IVP solution is not itself periodic.

Question 7 – Solution

Strategy. Integrate the forcing after multiplying by e^{2x} , then separate periodic and decaying terms.

Step 1: Integrate. The equation becomes

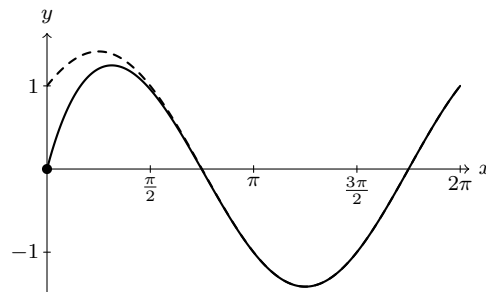
$$(e^{2x}y)' = e^{2x}(3\cos x + \sin x).$$

Seek an antiderivative $e^{2x}(A\cos x + B\sin x)$. Matching coefficients gives $2A + B = 3$ and $2B - A = 1$, hence $A = B = 1$. Indeed,

$$[e^{2x}(\cos x + \sin x)]' = e^{2x}(3\cos x + \sin x).$$

Therefore $y = \cos x + \sin x + Ce^{-2x}$. Since $y(0) = 0$, $C = -1$, giving

$$y = \cos x + \sin x - e^{-2x}.$$



Solid: IVP solution. Dashed: periodic solution $y_p = \cos x + \sin x$.

Step 2: The only periodic member. The function $y_p = \cos x + \sin x$ is 2π -periodic and satisfies $y_p' + 2y_p = 3\cos x + \sin x$. For the general solution,

$$y(x + 2\pi) - y(x) = Ce^{-2x}(e^{-4\pi} - 1).$$

This vanishes for every x only if $C = 0$. Hence $y_p = \cos x + \sin x$ is the unique periodic member; the IVP has $C = -1$ and is not periodic.

Step 3: Uniform accuracy after a time. The exact discrepancy is $|y - y_p| = e^{-2x}$, strictly decreasing on $x \geq 0$. Thus

$$e^{-2X} \leq \frac{1}{100} \iff X \geq \ln 10.$$

The earliest such time is $\ln 10$. The term Ce^{-2x} is transient because it vanishes as $x \rightarrow \infty$, leaving the periodic response.

Question 8

A switched forcing acts on a linear equation for $t \geq 0$:

$$y' + y = f(t), \quad y(0) = 0, \quad f(t) = \begin{cases} 2, & 0 \leq t < 1, \\ 0, & t > 1. \end{cases}$$

Seek a continuous function that is continuously differentiable on each side of $t = 1$ and satisfies the equation away from that switching time. No derivative at $t = 1$ is assumed.

Tasks

1. Solve on $0 \leq t < 1$ using an integrating factor, then use continuity to solve on $t > 1$.
2. Find the global maximum for $t \geq 0$, including its time, and the limit as $t \rightarrow \infty$.
3. Compute the left and right derivatives at $t = 1$. Is the resulting function a classical differentiable solution at that point?
4. Evaluate $\int_0^\infty y(t) dt$ and relate it to the total forcing $\int_0^\infty f(t) dt$. Sketch the continuous response in your solution, showing the change in slope at the switch.

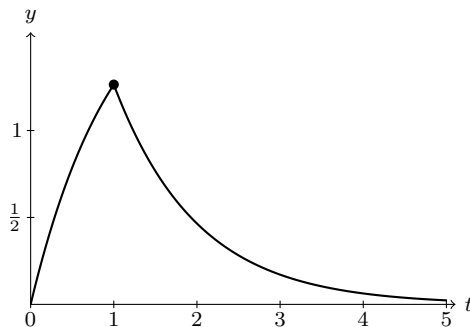
Question 8 – Solution

Strategy. Integrate separately on the two smooth pieces and match values at the switch; continuity need not imply a continuous derivative.

Step 1: Solve and match. Before the switch, $(e^t y)' = 2e^t$, so $y = 2 + Ce^{-t}$. Initial data give $C = -2$ and $y(1) = A := 2(1 - e^{-1})$. Afterward $(e^t y)' = 0$; matching this value yields

$$y(t) = \begin{cases} 2(1 - e^{-t}), & 0 \leq t \leq 1, \\ Ae^{-(t-1)}, & t > 1, \end{cases} \quad A = 2(1 - e^{-1}).$$

Differentiation on each open piece verifies its corresponding equation.



The maximum occurs at the switch $t = 1$, where the derivative jumps by -2 .

Step 2: Maximum and derivative jump. Before 1, $y' = 2e^{-t} > 0$; after 1, $y' = -Ae^{-(t-1)} < 0$. Hence the global maximum is A at $t = 1$, and the limit is zero. At the switch,

$$y'(1^-) = 2/e, \quad y'(1^+) = -A = 2/e - 2.$$

The jump is -2 , so the derivative does not exist there. The response meets the requested piecewise conditions, but is not a classical differentiable solution at the switch.

Step 3: Total response. Direct integration gives

$$\int_0^1 2(1 - e^{-t}) dt = 2/e, \quad \int_1^\infty Ae^{-(t-1)} dt = A,$$

so $\int_0^\infty y(t) dt = 2 = \int_0^\infty f(t) dt$. Equivalently, integrate $y' + y = f$ on the two pieces: continuity cancels the joining values and $y(0) = \lim_{t \rightarrow \infty} y = 0$ cancels the endpoint contribution.

Question 9

The equation

$$xy' - y = x^2$$

has a vanishing leading coefficient at $x = 0$. Initially impose $y(1) = 2$.

Tasks

1. Solve the IVP on $x > 0$ using an integrating factor after normalization.
2. Determine whether this solution extends to a differentiable solution of the original equation on all of $\mathbb{R}$. If so, prove that such an extension is unique.
3. Now discard $y(1) = 2$ and prescribe only $y(0) = 0$. Determine all differentiable solutions on $\mathbb{R}$ satisfying this new initial condition.
4. Decide whether a differentiable solution with $y(0) = 1$ can exist. Explain why dividing by x and invoking a regular standard-form IVP at 0 would be invalid.

Question 9 – Solution

Strategy. Normalize only where $x \neq 0$. To cross the exceptional point, return to the original equation and check value and derivative matching.

Step 1: Solve on the positive half-line. Dividing by $x > 0$ gives $y' - y/x = x$. The integrating factor is $e^{-\ln x} = 1/x$, so

$$(y/x)' = 1, \quad y = x^2 + Cx.$$

Since $y(1) = 2$, $C = 1$, giving $y = x^2 + x \quad (x > 0)$.

Step 2: Unique differentiable extension. The polynomial $x^2 + x$ satisfies $xy' - y = x^2$ everywhere, including $x = 0$. Any solution on $x < 0$ also satisfies $(y/x)' = 1$, hence has form $x^2 + Dx$. Continuity at 0 forces $y(0) = 0$, and differentiability requires the slopes at zero to match: $D = 1$. Thus

$$y = x^2 + x \text{ is the unique differentiable extension to } \mathbb{R}.$$

Step 3: Data only at the exceptional point. With only $y(0) = 0$, the same argument allows any common slope C on the two sides. All global differentiable solutions are

$$y = x^2 + Cx, \quad C \in \mathbb{R}.$$

Indeed $x(2x + C) - (x^2 + Cx) = x^2$, and every such polynomial has value zero at 0. This IVP is therefore nonunique.

Step 4: Incompatible data and normalization. At $x = 0$, the original equation requires $-y(0) = 0$, ruling out $y(0) = 1$. The normalized coefficient $-1/x$ is undefined at zero, so regular standard-form conclusions cannot be applied there. The algebraic condition at the exceptional point must be checked separately.

Question 10

Let a be a real parameter in

$$y' + ay = e^{-x}, \quad y(0) = 1.$$

Tasks

1. Solve the IVP for every real a , handling separately any parameter value for which the usual antiderivative formula changes.
2. Verify the initial condition and equation, and state the solution interval for each parameter.
3. For fixed x , show that the formula for $a \neq 1$ tends to the $a = 1$ solution as $a \rightarrow 1$. Explain why an apparent denominator singularity in a parameter need not be a singularity of the solution.
4. Determine exactly which values of a produce a bounded solution on $[0, \infty)$, and compute the limit or divergence as $x \rightarrow \infty$ in each parameter range.

Question 10 – Solution

Strategy. Keep the parameter visible during integration, and isolate the exceptional exponent before dividing by $a - 1$.

Step 1: Integrate with the initial value. With $\mu = e^{ax}$,

$$(e^{ax}y)' = e^{(a-1)x}, \quad e^{ax}y = 1 + \int_0^x e^{(a-1)t} dt.$$

Consequently

$$y(x) = \begin{cases} \frac{e^{-x} + (a-2)e^{-ax}}{a-1}, & a \neq 1, \\ (1+x)e^{-x}, & a = 1. \end{cases}$$

Each formula is defined on $\mathbb{R}$.

Step 2: Verify. At zero the numerator for $a \neq 1$ is $a - 1$, so $y(0) = 1$; the exceptional formula also gives 1. Differentiating the definite-integral identity above gives $y' + ay = e^{-x}$. At $a = 1$, a direct check is $y' = -xe^{-x}$ and $y' + y = e^{-x}$.

Step 3: Remove the apparent parameter singularity. For $a \neq 1$, rewrite the result as

$$y = e^{-ax} \left[1 + \frac{e^{(a-1)x} - 1}{a-1} \right].$$

For fixed x , the quotient tends to x as $a \rightarrow 1$, by the derivative of the exponential (also at $x = 0$). Hence $y \rightarrow (1+x)e^{-x}$. The singularity in the divided parameter formula is removable.

Step 4: Long-time behavior. If $a > 0$ and $a \neq 1$, both exponentials decay, so $y \rightarrow 0$; the same holds at $a = 1$. If $a = 0$, $y = 2 - e^{-x} \rightarrow 2$. If $a < 0$, the coefficient $(a-2)/(a-1)$ is positive and e^{-ax} grows without bound, so $y \rightarrow +\infty$. Thus

$$y \text{ is bounded on } [0, \infty) \iff a \geq 0.$$

Question 1

Consider the initial-value problem

$$y' = 2x(1 + y^2), \quad y(0) = 1.$$

Tasks

1. Solve by separation, explaining how the initial condition selects the integration constant.
2. Find the maximal open interval containing 0 on which the solution is finite and differentiable.
3. Determine its monotonicity, minimum, and behavior at both interval endpoints.
4. A student says that the periodicity of tangent extends this same solution through each vertical asymptote. Explain the error and verify the valid solution directly.

Question 1 – Solution

Strategy. Separate using the everywhere-positive factor $1 + y^2$, then respect the range of the inverse tangent.

Step 1: Integrate and select the branch.

$$\frac{dy}{1+y^2} = 2x \, dx, \quad \arctan y = x^2 + C, \quad C = \frac{\pi}{4}.$$

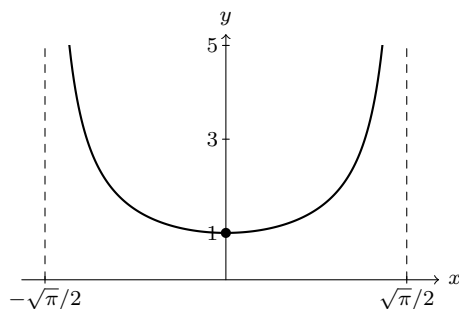
Thus $y(x) = \tan(x^2 + \pi/4)$ near 0. No constant solution was lost, since $1 + y^2$ never vanishes for real y .

Step 2: Identify the interval and shape. For the branch through 0, we need $-\pi/2 < x^2 + \pi/4 < \pi/2$. Hence

$$I = (-\sqrt{\pi}/2, \sqrt{\pi}/2).$$

The derivative has the sign of x : the solution decreases to its minimum $y(0) = 1$ and then increases. At either endpoint approached from inside I , the angle tends to $\pi/2$ from below, so $y \rightarrow +\infty$.

Step 3: Verify and explain the obstruction. Differentiation gives $y' = 2x \sec^2(x^2 + \pi/4) = 2x(1 + y^2)$, and $y(0) = 1$. A real differentiable extension through an endpoint would have a finite, continuous value there, contradicting blow-up. Other tangent branches lie on separate intervals; periodicity does not connect them across a pole.



The branch through $(0, 1)$ has a pole at each dashed boundary.

Question 2

For the autonomous equation

$$y' = y(3 - y),$$

consider the initial condition $y(0) = 4$.

Tasks

1. Find every constant solution before dividing by any expression involving y .
2. Obtain the nonconstant solution family by separation and solve the given initial-value problem.
3. Find its maximal interval containing 0, its direction of motion, and its limiting values at the ends of that interval.
4. Find the exact positive time when $y = 7/2$. Explain why this solution never reaches or crosses $y = 3$ at a finite time.

Question 2 – Solution

Strategy. Preserve the two equilibrium solutions, then use partial fractions on a nonconstant branch.

Step 1: Separate without losing constants. The constant solutions are $y \equiv 0, y \equiv 3$. Away from those values,

$$\frac{1}{y(3-y)} = \frac{1}{3y} + \frac{1}{3(3-y)}, \quad \frac{1}{3} \ln \left| \frac{y}{3-y} \right| = x + C.$$

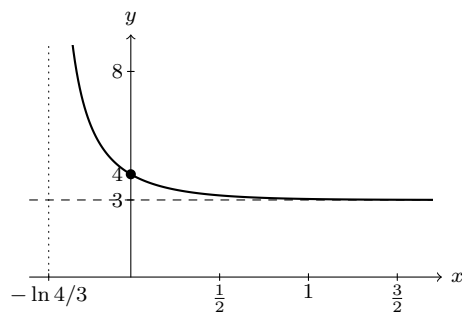
The nonconstant family can be written $y = 3/(1 + Ae^{-3x})$, $A \neq 0$, on intervals where the denominator is nonzero. The value $A = 0$ restores $y \equiv 3$; $y \equiv 0$ must still be listed separately.

Step 2: Apply the data and determine the domain. From $4 = 3/(1 + A)$, $A = -1/4$, so

$$y(x) = \frac{3}{1 - \frac{1}{4}e^{-3x}}, \quad I = (-\ln 4/3, \infty).$$

Here $y > 3$, so $y' < 0$. The limits are $+\infty$ at the left endpoint and 3 as $x \rightarrow \infty$. Writing $D = 1 - \frac{1}{4}e^{-3x}$ gives $y' = -9e^{-3x}/(4D^2) = y(3-y)$, verifying the equation.

Step 3: Compute the passage time. The equation $7/2 = 3/(1 - \frac{1}{4}e^{-3x})$ gives $x = \frac{1}{3} \ln(7/4)$. At any finite $x \in I$, the exponential is positive, so $y - 3 > 0$: the limiting level is never attained or crossed.



The decreasing solution stays above the dashed equilibrium $y = 3$.

Question 3

Consider an equation whose right-hand side is undefined on the line $y = 2$:

$$y' = -\frac{e^x}{y-2}, \quad y(0) = 1.$$

Tasks

1. Separate the equation and obtain an implicit relation.
2. Select the correct explicit branch and determine its maximal interval containing 0.
3. Find the limiting value of y and the behavior of y' at the finite endpoint. Decide whether that endpoint can be included.
4. Another student takes the opposite square-root sign, then proposes switching signs at the endpoint. Test both claims against the equation and initial condition.

Question 3 – Solution

Strategy. Integration gives a squared relation; the initial condition and the excluded line determine which part is a solution.

Step 1: Integrate and choose the sign.

$$(y - 2) dy = -e^x dx, \quad \frac{1}{2}(y - 2)^2 = -e^x + C.$$

At $(0, 1)$, $C = 3/2$, so $(y - 2)^2 = 3 - 2e^x$. Since $y(0) - 2 = -1$, continuity selects

$$y = 2 - \sqrt{3 - 2e^x}, \quad I = (-\infty, \ln(3/2)).$$

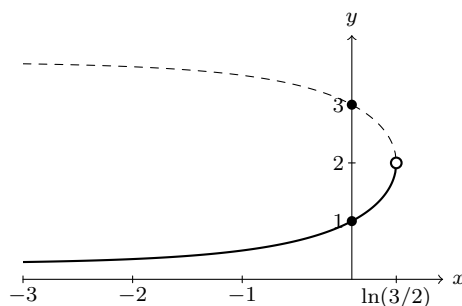
The radicand must be strictly positive: zero would give the forbidden value $y = 2$.

Step 2: Check behavior and verify. Throughout I ,

$$y' = \frac{e^x}{\sqrt{3 - 2e^x}} = -\frac{e^x}{y - 2} > 0.$$

As $x \rightarrow -\infty$, $y \rightarrow 2 - \sqrt{3}$. As $x \uparrow \ln(3/2)$, $y \rightarrow 2$ but $y' \rightarrow +\infty$. A finite limit of y alone does not permit extension: the equation is undefined at the endpoint and a finite derivative is impossible there.

Step 3: Audit the other branch. The branch $2 + \sqrt{3 - 2e^x}$ satisfies the differential equation on I , but has $y(0) = 3$, not 1. Switching signs would still pass through $y = 2$, and for larger x the radicand is negative. Neither proposal extends the given real solution.



Solid: $y(0) = 1$. Dashed: $y(0) = 3$. The common limiting point is excluded.

Question 4

An explicit elementary formula is not required in this problem:

$$y' = \frac{x}{1 + e^y}, \quad y(0) = 0.$$

Tasks

1. Use separation to find an implicit equation for the solution.
2. Prove that it determines exactly one real value of y for every real x , and that the resulting function is differentiable.
3. Determine its symmetry, monotonicity, and minimum. Prove the bound $0 \leq y(x) \leq x^2/4$ and identify every equality case.
4. Verify the initial condition and differential equation, and explain why failure to isolate y in elementary functions is not a failure to solve the problem.

Question 4 – Solution

Strategy. Treat the left side of the integrated equation as an invertible function rather than forcing an elementary inverse.

Step 1: Integrate. Multiplication and integration give

$$(1 + e^y) dy = x dx, \quad \boxed{y + e^y = 1 + \frac{x^2}{2}}.$$

The constant is 1 because $y(0) = 0$.

Step 2: Prove global existence and uniqueness of the implicit value. Let $F(u) = u + e^u$. It is continuous and strictly increasing because $F'(u) = 1 + e^u > 0$. Also $F(u) \rightarrow -\infty$ as $u \rightarrow -\infty$ and $F(u) \rightarrow \infty$ as $u \rightarrow \infty$. It therefore has an inverse on all of $\mathbb{R}$. Since its derivative never vanishes, the inverse is differentiable. Thus

$$y(x) = F^{-1}(1 + x^2/2)$$

defines a unique differentiable solution for every real x . Any solution through $(0, 0)$ must satisfy the same integrated identity, establishing uniqueness for the initial-value problem.

Step 3: Extract shape and an exact bound. Uniqueness of the implicit value and the dependence on x^2 show $y(-x) = y(x)$. Since $F(0) = 1$, $y \geq 0$, with equality only at $x = 0$. The differential equation makes y decreasing for $x < 0$ and increasing for $x > 0$, so its unique minimum is 0.

Using $e^y \geq 1 + y$, with equality only at $y = 0$, gives

$$1 + \frac{x^2}{2} = y + e^y \geq 1 + 2y, \quad \boxed{0 \leq y \leq \frac{x^2}{4}}.$$

Equality in either bound occurs only at $x = 0$.

Step 4: Verify and interpret. Implicit differentiation yields $(1 + e^y)y' = x$, exactly the given equation. At $x = 0$, strict monotonicity of F forces $y = 0$. The implicit formula uniquely specifies every solution value, its full domain, and its derivative; an elementary inverse is unnecessary.

Question 5

Consider these two equations for real x and y :

$$(A) \quad y' = x + y, \quad (B) \quad y' = xy + x + y + 1.$$

Here “directly separable” means that the right-hand side can be written $g(x)h(y)$ on the region in question, without a change of variables.

Tasks

1. Prove that any product $f(x, y) = g(x)h(y)$ satisfies

$$f(a, c)f(b, d) = f(a, d)f(b, c).$$

2. Use $a = c = 1$ and $b = d = 2$ to show that (A) is not directly separable on the rectangle $[1, 2] \times [1, 2]$.
3. Factor the right-hand side of (B), then find all its real solutions on intervals and list any constant solution separately.
4. Solve (B) with $y(0) = -2$, verify it, and explain why division by $x + 1$ is unnecessary even at $x = -1$.

Question 5 – Solution

Strategy. Test whether variables actually factor before performing separation; a sum of terms can still conceal a product.

Step 1: Establish the necessary product test. If $f(x, y) = g(x)h(y)$, then both products in the stated identity equal $g(a)g(b)h(c)h(d)$. This argument does not divide by any factor and remains valid if some factors vanish.

Step 2: Reject direct separation for (A). For $f(x, y) = x + y$, the proposed rectangle gives

$$f(1, 1)f(2, 2) = 2 \cdot 4 = 8, \quad f(1, 2)f(2, 1) = 3 \cdot 3 = 9.$$

The necessary identity fails, so there is no product representation on this rectangle. This conclusion concerns direct separation on that region; it makes no claim about other methods of solving (A).

Step 3: Solve the factored equation (B). Here $xy + x + y + 1 = (x + 1)(y + 1)$. First, $y \equiv -1$ is a constant solution. For $y \neq -1$,

$$\frac{dy}{y + 1} = (x + 1) dx, \quad \ln |y + 1| = \frac{x^2}{2} + x + C.$$

Absorbing the fixed sign into a nonzero constant gives

$$\boxed{y = -1 + Ae^{x^2/2+x}}.$$

Allowing $A = 0$ includes the constant solution as well. These are all solutions: for any differentiable solution, the product rule applied to $(y + 1)e^{-x^2/2-x}$ gives derivative zero, even at $y = -1$.

Step 4: Apply and check the data. The initial condition requires $A = -1$, so $\boxed{y = -1 - e^{x^2/2+x}}$ on all of $\mathbb{R}$. Its derivative is $-(x + 1)e^{x^2/2+x} = (x + 1)(y + 1)$, and $y(0) = -2$. Separation integrates the factor $x + 1$; it never requires its reciprocal. The point $x = -1$ is an ordinary point with $y' = 0$.

Question 6

A nonnegative quantity satisfies

$$y' = -3y^{2/3}, \quad y(0) = 8, \quad t \geq 0,$$

where the prime means d/dt . For real y , define $y^{2/3} = (\sqrt[3]{y})^2$.

Tasks

1. Solve while $y > 0$ and find the first time the quantity reaches zero.
2. Find and justify the unique continuation for all $t \geq 0$ if the solution must remain nonnegative.
3. If negative values are allowed, classify all continuously differentiable continuations after the first zero, including solutions that wait at zero before becoming negative.
4. Check the equation at every joining time and identify exactly what is lost by dividing by $y^{2/3}$.

Question 6 – Solution

Strategy. Separate only where $y \neq 0$, then use the sign of y' and check any joins at zero directly.

Step 1: Reach zero. While $y > 0$, integration gives $3y^{1/3} = -3t + C$. The data yield $y = (2 - t)^3$ for $0 \leq t < 2$, so the first zero is $t = 2$.

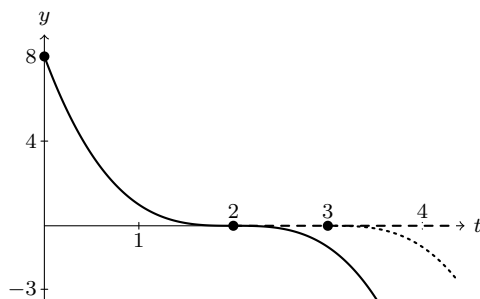
Step 2: Enforce nonnegativity. The equation always gives $y' \leq 0$. Once a nonnegative solution reaches zero it cannot increase or decrease without violating that constraint. Thus the unique nonnegative continuation is $(2 - t)^3$ up to 2, followed by 0 forever.

Step 3: Classify real continuations. For any finite $b \geq 2$, a real continuation is

$$y_b(t) = \begin{cases} (2 - t)^3, & 0 \leq t \leq 2, \\ 0, & 2 \leq t \leq b, \\ -(t - b)^3, & t \geq b. \end{cases}$$

Also allow $b = \infty$, meaning the solution stays zero forever. These exhaust the possibilities: monotonicity makes the zero set after 2 an interval; once negative, separation gives $\sqrt[3]{y} = b - t$, where continuity fixes the constant at the departure time b .

Step 4: Verify the joins and the lost solutions. Each cubic piece has $y' = -3(\sqrt[3]{y})^2$. At $t = 2$ and any finite b , both one-sided derivatives are 0, matching the equation at $y = 0$. The pieces are therefore C^1 . Division excludes zero itself, so it misses the constant zero solution and all zero waiting intervals; using one integration constant through zero is unjustified.



After $t = 2$: solid $b = 2$, dotted $b = 3$, dashed $b = \infty$ (zero forever).

Question 7

A positive concentration obeys a power-law decay model

$$\frac{dy}{dt} = -ky^p, \quad k > 0, \quad p \in \mathbb{R}.$$

Use fixed units for time and concentration. Exact observations are

$$y(0) = 16, \quad y(1) = 8, \quad y(2) = 4.$$

Assume one pair of constants (k, p) describes the entire observation interval.

Tasks

1. Express each of the two observed one-unit decay times as a separated definite integral, without assuming $p = 1$.
2. Prove that these observations determine p uniquely. Treat the case $p = 1$ without dividing by $1 - p$.
3. Find k and the resulting solution, and predict when the concentration first reaches 1.
4. Explain why the first two observations alone would not determine p .

Question 7 – Solution

Strategy. Compare passage-time integrals on two concentration intervals related by scaling; the unknown constant k then cancels.

Step 1: Write the two passage times. Since $y > 0$ and $k > 0$, the concentration decreases strictly. Separation gives

$$1 = \frac{1}{k} \int_8^{16} s^{-p} ds, \quad 1 = \frac{1}{k} \int_4^8 s^{-p} ds.$$

These integrals are positive and finite for every real p .

Step 2: Recover the exponent. In the first integral put $s = 2u$:

$$\int_8^{16} s^{-p} ds = 2^{1-p} \int_4^8 u^{-p} du.$$

The observed equal passage times force $2^{1-p} = 1$, hence $\boxed{p = 1}$. Conversely, $p = 1$ makes the two integrals equal. No division by $1 - p$ was used, so the logarithmic case was retained throughout.

Step 3: Determine the rate and prediction. For $p = 1$,

$$k = \int_8^{16} \frac{ds}{s} = \ln 2, \quad \ln \frac{y}{16} = -t \ln 2.$$

Therefore $\boxed{y(t) = 16 2^{-t}}$. It gives exactly 16, 8, 4 at the measured times, and differentiating gives $y' = -(\ln 2)y$. It is positive for every real t . The value 1 is first reached at $\boxed{t = 4}$ because the solution is strictly decreasing.

Step 4: Identify the information supplied by the third observation. With only $y(0) = 16$ and $y(1) = 8$, each real p could be paired with

$$k(p) = \int_8^{16} s^{-p} ds > 0.$$

The corresponding separated solution decreases from 16 to 8 in one unit of time and stays positive during that interval. Thus infinitely many exponents fit the first two data; the second measured halving time identifies the exponent within this model family.

Question 8

Consider the parameterized initial-value problem

$$y' = (\sin x)y^2, \quad y(0) = a, \quad a > 0.$$

Tasks

1. Solve by separation and determine the maximal open interval containing 0 for every positive a .
2. Find the precise threshold separating global bounded solutions from finite-endpoint blow-up. Include the threshold case.
3. For the global solutions, find the minimum, maximum, and a period.
4. A student argues that $\int_0^{2\pi} \sin x \, dx = 0$ guarantees $y(2\pi) = a$ for every $a > 0$. Explain exactly when this reasoning is valid and why it can fail.

Question 8 – Solution

Strategy. Track the denominator of the separated formula throughout an interval, not merely at its endpoints.

Step 1: Integrate. Since $a > 0$, separation is valid near 0:

$$-\frac{1}{y} = -\cos x + C, \quad \boxed{y = \frac{a}{1 - a + a \cos x}}.$$

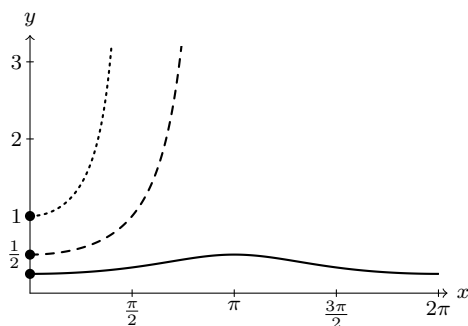
Its derivative is $a^2 \sin x / (1 - a + a \cos x)^2 = (\sin x)y^2$, and $y(0) = a$.

Step 2: Classify all positive parameters. The denominator $D = 1 - a + a \cos x$ ranges from $1 - 2a$ to 1.

- If $0 < a < 1/2$, then $D > 0$ everywhere: $I = \mathbb{R}$ and y is bounded.
- If $a = 1/2$, then $y = 1/(1 + \cos x)$ and $I = (-\pi, \pi)$.
- If $a > 1/2$, let $\alpha = \arccos(1 - 1/a) \in (0, \pi)$. Then $I = (-\alpha, \alpha)$.

In the last two cases $D \downarrow 0$ at either endpoint from inside I , so $y \rightarrow +\infty$. Thus the threshold itself already blows up.

Step 3: Determine extrema and audit the return claim. For $0 < a < 1/2$, $\min y = a$ at $x = 2n\pi$ and $\max y = a/(1 - 2a)$ at $x = (2n + 1)\pi$, $n \in \mathbb{Z}$; 2π is a period. Only this parameter range gives a solution through the full cycle. For $a \geq 1/2$, a pole occurs before reaching 2π . Substituting 2π into the algebraic formula gives a value on a disconnected branch, not a continuation of the initial solution.



Solid: $a = 1/4$. Dashed: $a = 1/2$. Dotted: $a = 1$. Only the solid curve completes a cycle.

Question 9

Let A and B be prescribed real numbers. Seek a real function y that is continuously differentiable on $[-1, 1]$ and satisfies

$$y' = 2xy^2, \quad y(-1) = A, \quad y(1) = B.$$

At the endpoints, derivatives are understood as one-sided derivatives.

Tasks

1. For $A \neq 0$, separate the equation and express the candidate solution in terms of A . Also treat $A = 0$.
2. Determine all pairs (A, B) for which a solution exists on the entire interval, and prove your conditions are sufficient as well as necessary.
3. Show why matching the two endpoint values in a formula does not by itself prove existence. Use $A = B = -2$ as a test case.
4. For every admissible pair, find $y(0)$ and decide whether the solution is unique.

Question 9 – Solution

Strategy. Separate from the left endpoint, then check the denominator on the whole prescribed interval before accepting the right endpoint.

Step 1: Obtain the candidate. For a nonzero solution,

$$-\frac{1}{y} = x^2 + C, \quad \frac{1}{y} = \frac{1}{A} + 1 - x^2, \quad \boxed{y = \frac{A}{1 + A(1 - x^2)}}.$$

For $A = 0$, $y \equiv 0$ is a solution. No solution can leave or cross zero: the right-hand side $2xy^2$ has a continuous y -derivative, $4xy$, so the usual local uniqueness theorem applies at every finite point. Consequently the separated formula applies to any solution with $A \neq 0$ throughout its interval.

Step 2: Test the entire interval. On $[-1, 1]$, the quantity $1 - x^2$ ranges over $[0, 1]$. If $A \geq 0$, the denominator is at least 1. If $-1 < A < 0$, it is at least $1 + A > 0$. For $A = -1$ it vanishes at $x = 0$; for $A < -1$ it vanishes at

$$x = \pm\sqrt{1 + 1/A} \in (-1, 1).$$

The nonzero numerator cannot cancel these zeros. Any admissible formula gives $y(1) = A$, so the necessary conditions are

$$\boxed{B = A \quad \text{and} \quad A > -1}.$$

They are also sufficient: the displayed formula, including $A = 0$, is smooth on a neighborhood of $[-1, 1]$. Its derivative is $2xA^2/[1 + A(1 - x^2)]^2 = 2xy^2$, and it gives both specified endpoint values.

Step 3: Expose the false endpoint test. For $A = B = -2$, the candidate is $y = -2/(2x^2 - 1)$. It has the desired values at $x = \pm 1$ but poles at $x = \pm 1/\sqrt{2}$. Hence it is not a solution on the entire interval.

Step 4: Give the center value and uniqueness. For every admissible pair, $\boxed{y(0) = A/(1 + A)}$. The initial value at -1 already fixes the candidate uniquely; for $A = 0$, local uniqueness also forbids any departure from zero. The second endpoint is a compatibility condition, not an extra free integration constant.

Question 10

Design an affine coefficient $g(x) = \alpha + \beta x$ in the separable equation

$$y' = g(x)(1 - y^2)$$

so that one solution satisfies

$$y(0) = 0, \quad y(1) = \frac{1}{2}, \quad y'(1) = 0.$$

The constants α and β are real.

Tasks

1. Find α and β and justify why the three requirements determine them uniquely.
2. Find an explicit formula for the selected solution. You may use exponentials instead of hyperbolic functions.
3. Determine its maximal interval, every zero, its global maximum, and its limits as $x \rightarrow \pm\infty$.
4. Verify the three requirements directly and list the constant solutions omitted when separating by $1 - y^2$.

Question 10 – Solution

Strategy. Use the zero terminal slope to constrain the coefficient, then use the separated integral to enforce the target height.

Step 1: Determine the coefficient. Since $1 - y(1)^2 = 3/4$, $y'(1) = 0$ forces $\alpha + \beta = 0$. The smooth right-hand side gives local uniqueness, so the solution starting at 0 cannot meet either constant solution $y = \pm 1$. On its branch $-1 < y < 1$,

$$\frac{1}{2} \ln \frac{1+y}{1-y} = \alpha x + \frac{\beta x^2}{2}.$$

At $x = 1$, this becomes $\frac{1}{2} \ln 3 = \alpha + \beta/2$. Hence

$$\boxed{\alpha = \ln 3, \quad \beta = -\ln 3}.$$

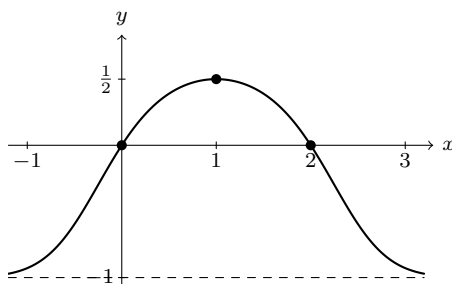
These two independent linear conditions have exactly one pair of coefficients.

Step 2: Solve and interpret the curve. Set $E(x) = 3^{2x-x^2} > 0$. Exponentiating and isolating y gives

$$\boxed{y(x) = \frac{E(x) - 1}{E(x) + 1}, \quad I = \mathbb{R}}.$$

Since $E + 1 > 0$, the formula is globally smooth and lies strictly between -1 and 1 . Its zeros are $x = 0, 2$. The exponent $2x - x^2 = 1 - (x - 1)^2$ has its unique maximum at 1 , and $(E - 1)/(E + 1)$ increases with E . Thus the unique global maximum is $y(1) = 1/2$, while both limits at infinity are -1 .

Step 3: Verify and restore constants. Here $E' = 2(\ln 3)(1 - x)E$ and $1 - y^2 = 4E/(E + 1)^2$, so $y' = (\ln 3)(1 - x)(1 - y^2)$. Also $E(0) = 1$ and $E(1) = 3$ verify both values and the zero slope. The omitted constants are $\boxed{y \equiv 1, y \equiv -1}$; neither satisfies $y(0) = 0$.



The unique maximum is $(1, 1/2)$; both tails approach the dashed level $y = -1$.

Question 1

Consider the differential equation

$$(2xy + 1) dx + (x^2 + y^2) dy = 0, \quad y(1) = 1.$$

A potential for $M dx + N dy$ is a function F satisfying $F_x = M$ and $F_y = N$; its level curves $F = C$ give implicit solutions.

Tasks

1. Check exactness and construct a potential, retaining the unknown function that arises when integrating with respect to x .
2. Apply the initial condition and verify the resulting implicit solution by differentiation.
3. Find the slope and tangent line at $(1, 1)$.
4. Prove that the selected level determines a unique differentiable function $y(x)$ on all of $\mathbb{R}$, despite the vanishing of $x^2 + y^2$ at the origin.

Question 1 – Solution

Strategy. Recover a potential from both coefficients, then examine its derivative with respect to y on the selected level.

Step 1: Test and integrate. Here $M_y = 2x = N_x$, so the form is exact on $\mathbb{R}^2$. Integrating $F_x = M$ gives

$$F = x^2y + x + h(y), \quad F_y = x^2 + h'(y).$$

Matching N requires $h'(y) = y^2$, so we may choose $F = x^2y + x + y^3/3$. At $(1, 1)$, $F = 7/3$. Thus

$$\boxed{x^2y + x + \frac{y^3}{3} = \frac{7}{3}}.$$

Step 2: Verify and find the tangent. Differentiating this identity gives

$$2xy + 1 + (x^2 + y^2)y' = 0,$$

exactly the original equation for a graph. At $(1, 1)$ this yields $y' = -3/2$, hence $\boxed{y - 1 = -\frac{3}{2}(x - 1)}$.

Step 3: Establish a global graph. For fixed x , the function $u \mapsto x^2u + u^3/3$ is strictly increasing and ranges over all real numbers. This remains true at $x = 0$, where the cubic alone is strictly increasing. Therefore the equation

$$x^2u + \frac{u^3}{3} = \frac{7}{3} - x$$

has exactly one real root for every real x .

The derivative $F_y = x^2 + y^2$ can vanish only at $(0, 0)$, but that point has $F = 0$, not $7/3$. It is consequently nonzero everywhere on the selected level. The implicit function theorem gives a differentiable local graph at each point; the uniqueness of each root makes those graphs agree. They form one differentiable solution on all of $\mathbb{R}$. The isolated degeneracy on a different level does not obstruct this IVP.

Question 2

A student studies

$$(e^y + y^2) dx + (xe^y + 2xy + \cos y) dy = 0, \quad y(0) = 0.$$

After integrating the coefficient of dx , the student writes

$$F_*(x, y) = xe^y + xy^2$$

and claims that $F_* = 0$ solves the initial-value problem.

Tasks

1. Check exactness and identify precisely what is missing from the student's integration.
2. Construct a correct potential and select the correct level using the data.
3. Compute the local slope at $(0, 0)$ and explain why the student's proposed level cannot be the required solution graph.
4. Integrate the coefficient of dy first as an independent check. Explain why the two correct potentials can differ by a constant.

Question 2 – Solution

Strategy. A partial antiderivative permits an arbitrary function of the other variable; matching the second coefficient determines it.

Step 1: Locate the missing term. Both cross partials equal $e^y + 2y$, so the form is exact on $\mathbb{R}^2$. Integration with respect to x must give

$$F = xe^y + xy^2 + h(y).$$

Then $F_y = xe^y + 2xy + h'(y)$, so $h'(y) = \cos y$. We may take $h(y) = \sin y$; treating it as merely a numerical constant would omit a required derivative.

Step 2: Apply and test the initial condition. The correct potential is $F = xe^y + xy^2 + \sin y$, and $F(0, 0) = 0$. Thus

$$\boxed{x(e^y + y^2) + \sin y = 0}.$$

Its derivative along a graph is exactly the given equation. At the initial point, $M = 1$ and $N = 1$, so the local graph exists and $\boxed{y'(0) = -1}$.

In contrast, $F_* = 0$ says $x(e^y + y^2) = 0$. Since $e^y + y^2 > 0$, this is the vertical line $x = 0$, which cannot define $y(x)$ on an open interval. Also $(F_*)_y = N - \cos y$, so its differential is not the stated form.

Step 3: Check by reversing the integration order. Integrating N with respect to y gives

$$F = xe^y + xy^2 + \sin y + g(x).$$

Now $F_x = e^y + y^2 + g'(x) = M$ requires $g'(x) = 0$, leaving only an additive constant. Any two potentials for the same form have difference with both partial derivatives zero; on the connected plane that difference is constant. Changing this constant only relabels the same solution levels.

Question 3

Let $a, b \in \mathbb{R}$. Consider

$$(3x^2y^2 + 2bxy + e^y) dx + (2ax^3y + 5x^2 + xe^y + 4y^3) dy = 0.$$

Tasks

1. Determine every pair (a, b) for which the form is exact on $\mathbb{R}^2$. Explain why testing equality at a single point is insufficient.
2. For those parameters, construct a potential and verify both of its first partial derivatives.
3. Find the implicit solution through $(0, 1)$ and its tangent line there.
4. A student checks $M_y = N_x$ only at $(0, 0)$ and accepts every pair (a, b) . Show why that check cannot distinguish any of the parameters.

Question 3 – Solution

Strategy. Exactness requires an identity on a region, so compare the variable-dependent terms before constructing a potential.

Step 1: Determine the parameters. The cross partials are

$$M_y = 6x^2y + 2bx + e^y, \quad N_x = 6ax^2y + 10x + e^y.$$

Their difference is $6(1-a)x^2y + (2b-10)x$. For this to vanish for all (x, y) , first put $y = 0$ and $x = 1$, obtaining $b = 5$. Then put $x = y = 1$, obtaining $a = 1$. Conversely, those choices make the difference identically zero. Hence

$$\boxed{a = 1, \quad b = 5}.$$

Step 2: Construct and verify the potential. With these values, integrating M in x yields

$$F = x^3y^2 + 5x^2y + xe^y + h(y).$$

Matching F_y with N gives $h'(y) = 4y^3$. Thus choose

$$F = x^3y^2 + 5x^2y + xe^y + y^4.$$

Direct differentiation gives $F_x = 3x^2y^2 + 10xy + e^y = M$ and $F_y = 2x^3y + 5x^2 + xe^y + 4y^3 = N$.

Step 3: Apply the data and find the tangent. Since $F(0, 1) = 1$, the selected level is $\boxed{F(x, y) = 1}$. At that point, $N = 4 \neq 0$, so it determines a local solution graph. Its slope is $-M/N = -e/4$, giving

$$\boxed{y - 1 = -\frac{e}{4}x}.$$

Step 4: Explain the failed one-point test. At $(0, 0)$, both cross partials equal 1 for every a, b : all parameter-dependent terms vanish there. This agreement is necessary at that point but says nothing about agreement on a neighborhood. Exactness cannot be certified from that single sample.

Question 4

The coefficient $N(x, y)$ is unknown in

$$(2x + y) dx + N(x, y) dy = 0.$$

Assume N is continuously differentiable on $\mathbb{R}^2$, the form is exact there, and the calibration data are $N(0, y) = 2y$ for every real y .

Tasks

1. Recover N uniquely and construct a potential.
2. Select the solution through $(0, 1)$ and find an explicit formula for its graph near that point.
3. Determine the maximal open interval of this graph and its slope at $x = 0$.
4. Find the vertical-tangent points of the full selected level curve. Explain why the full curve is not a single differentiable function $y(x)$.

Question 4 – Solution

Strategy. Use exactness as an equation for the missing coefficient, then distinguish a complete level curve from its graph branches.

Step 1: Recover the form. Exactness gives $N_x = M_y = 1$, so $N = x + h(y)$. The data force $h(y) = 2y$, uniquely. A potential is

$$N = x + 2y, \quad F = x^2 + xy + y^2.$$

Its derivatives are $2x + y$ and $x + 2y$, as required.

Step 2: Select a branch. The initial point gives $F = 1$. Solving the quadratic in y and selecting $y(0) = 1$ yields

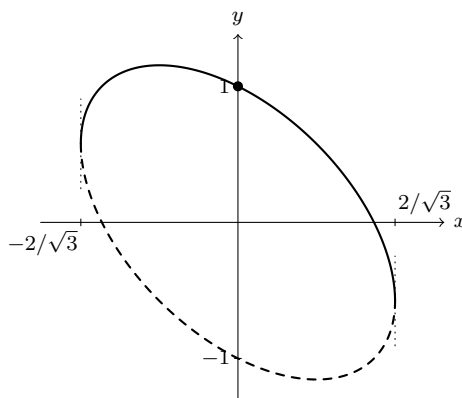
$$y = \frac{-x + \sqrt{4 - 3x^2}}{2}, \quad I = (-2/\sqrt{3}, 2/\sqrt{3}).$$

At $(0, 1)$, $y' = -(2x + y)/(x + 2y) = -1/2$.

Step 3: Interpret the endpoints and full curve. On this branch, $N = \sqrt{4 - 3x^2} > 0$. At the two endpoints, $N = 0$, while $M = 3x/2 \neq 0$. Thus a finite y' cannot satisfy $M + Ny' = 0$ there. The full ellipse has vertical tangents at

$$(2/\sqrt{3}, -1/\sqrt{3}), \quad (-2/\sqrt{3}, 1/\sqrt{3}).$$

It remains a smooth curve at these points, but cannot be continued through either as a differentiable graph over x . Most vertical lines through the ellipse meet both quadratic branches, so the complete ellipse is not one function $y(x)$.



Solid: IVP branch, excluding its vertical-tangent endpoints. Dashed: the other half.

Question 5

Consider the exact equation

$$-2x \, dx + 2y \, dy = 0, \quad y(0) = 0.$$

In this problem a solution means a real, continuously differentiable function $y(x)$ on an open interval containing 0, satisfying $-2x + 2yy' = 0$ at every point, including 0.

Tasks

1. Find a potential and the level selected by the initial condition.
2. Classify all solution graphs on such an interval. Justify why no additional sign-switching choices are differentiable at 0.
3. Check $y = x$, $y = -x$, $y = |x|$ and $y = -|x|$ against the stated definition.
4. Explain why neither division by y nor the usual nonzero- F_y implicit-function criterion settles the initial point.

Question 5 – Solution

Strategy. The selected level has crossing branches. Analyze differentiability at their intersection directly in the original equation.

Step 1: Find the level. Since $M_y = N_x = 0$, a potential is $F = y^2 - x^2$. Along any solution, $dF/dx = 0$, and the initial condition gives

$$\boxed{y^2 - x^2 = 0}.$$

Thus for each $x \neq 0$, y/x is either 1 or -1 .

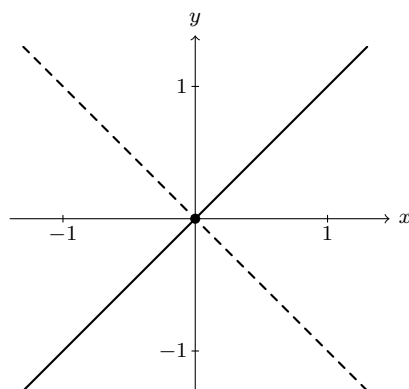
Step 2: Classify differentiable choices. Continuity makes y/x constant on each of the two connected pieces $x < 0$ and $x > 0$: a continuous function taking only the values ± 1 cannot switch between them. Write those constants as s_- and s_+ . At 0, the left and right difference quotients are s_- and s_+ . Differentiability requires them to agree. Therefore the only possibilities are

$$\boxed{y = x \quad \text{or} \quad y = -x}.$$

Both extend to all of $\mathbb{R}$.

Step 3: Verify and identify the failures. For either line, $yy' = x$ everywhere, including 0. The choices $|x|$ and $-|x|$ satisfy the level identity and the equation away from 0, but their one-sided derivatives disagree at 0. They fail the required definition.

Step 4: Explain the degenerate point. Dividing gives $y' = x/y$, which is undefined at the initial point and discards information about solutions of the original form there. Also $F_y = 2y = 0$ at $(0, 0)$, so the nonzero- F_y criterion does not apply. Its failure does not prove nonexistence: here two valid graph branches pass through the point.



Two differentiable solutions cross at $(0, 0)$: solid $y = x$, dashed $y = -x$.

Question 6

On the half-plane $x > 0$, consider

$$(2y + x) dx + x dy = 0, \quad y(1) = 2.$$

An integrating factor is a nonzero function that makes a differential form exact when multiplied into both coefficients.

Tasks

1. Show that the original form is not exact. Derive, using the product rule, the condition on an integrating factor $\mu(x)$ depending only on x .
2. Find every such nonzero factor on $x > 0$, choose one, and construct a potential for the multiplied form.
3. Solve and verify the initial-value problem on $x > 0$.
4. Now consider the original equation on the whole plane. Decide whether the selected solution can extend through $x = 0$, and identify which member of the implicit solution family does extend differentiably through that point.

Question 6 – Solution

Strategy. Derive the integrating-factor condition from exactness rather than importing a formula, and return to the original equation at a zero of the chosen factor.

Step 1: Derive the factor. Originally $M_y = 2$ while $N_x = 1$. For $\mu = \mu(x)$, exactness requires

$$(\mu M)_y = 2\mu = (\mu N)_x = x\mu' + \mu.$$

Thus $x\mu' = \mu$. On $x > 0$, its nonzero solutions are $\mu = Kx$, $K \neq 0$. Choose $\boxed{\mu = x}$, which is nonzero throughout the stated half-plane.

Step 2: Integrate the exact form. The multiplied coefficients are $2xy + x^2$ and x^2 . Integrating in y gives

$$F = x^2y + g(x), \quad F_x = 2xy + g'(x) = 2xy + x^2.$$

Hence $g = x^3/3$ is a choice, giving the family $\boxed{x^2y + x^3/3 = C}$.

Step 3: Apply and verify the data. At $(1, 2)$, $C = 7/3$, so

$$\boxed{y = \frac{7}{3x^2} - \frac{x}{3}, \quad x > 0}.$$

Its derivative is $-14/(3x^3) - 1/3$, and substitution gives

$$2y + x + xy' = \frac{14}{3x^2} - \frac{2x}{3} + x - \frac{14}{3x^2} - \frac{x}{3} = 0.$$

The initial value is also 2.

Step 4: Check extension in the original equation. The selected solution tends to $+\infty$ as $x \downarrow 0$, so it cannot extend continuously through 0. In the general family, $y = C/x^2 - x/3$, a finite extension requires $C = 0$. This member extends as $\boxed{y = -x/3}$ on all of $\mathbb{R}$ and satisfies the original equation even at 0, where the equation requires $2y(0) = 0$. The multiplied form alone gives $0 = 0$ there because its factor vanishes; that weakened condition must not replace the original one.

Question 7

Compare the following forms on $\mathbb{R}^2$:

(A) $x \, dx + y \, dy = 0$,

(B) $e^x x \, dx + e^x y \, dy = 0$,

(C) $(1 + x^2 + y^2)x \, dx + (1 + x^2 + y^2)y \, dy = 0$.

Tasks

1. Determine which forms are exact and give a potential for each exact form.
2. Prove that all three nevertheless have the same differentiable solution graphs.
3. Find their common solution graph with $y(0) = 2$ and its maximal open interval containing 0.
4. Let $dF = M \, dx + N \, dy$. Explain why multiplication by a continuous function $h(F)$ produces another exact form, and state an additional condition that guarantees equivalence of the graph equations.

Question 7 – Solution

Strategy. Separate two different questions: whether the displayed coefficients are partial derivatives of a potential, and whether multiplying the equation changes its solutions.

Step 1: Test each form. For (A), $M_y = N_x = 0$, with potential

$$F = \frac{x^2 + y^2}{2}.$$

For (B), $M_y = 0$ and $N_x = e^x y$, which are not equal on the plane, so (B) is not exact there. For (C), both cross partials are $2xy$, and a potential is

$$G = \frac{x^2 + y^2}{2} + \frac{(x^2 + y^2)^2}{4}.$$

Differentiating G gives the two coefficients in (C).

Step 2: Establish equivalence. The equations for a graph are respectively

$$x + yy' = 0, \quad e^x(x + yy') = 0, \quad (1 + x^2 + y^2)(x + yy') = 0.$$

Both multiplying factors are strictly positive at every real point. Each equation therefore holds exactly when the first does. Exactness of a particular form is not required for it to share the same solutions as an exact form.

Step 3: Select the common IVP solution. The first potential gives $x^2 + y^2 = 4$. Continuity from $y(0) = 2$ selects

$$\boxed{y = \sqrt{4 - x^2}, \quad I = (-2, 2)}.$$

Here $y' = -x/\sqrt{4 - x^2}$ verifies all three equations. At either endpoint, $y = 0$ but $x \neq 0$, so no finite derivative can satisfy $x + yy' = 0$; a differentiable graph cannot extend through it.

Step 4: Generalize using the chain rule. Choose an antiderivative H of the continuous function h . Then

$$d(H(F)) = H'(F) dF = h(F)(M dx + N dy).$$

This gives a potential for the new form. If $h(F)$ is nonzero throughout the region considered, division is valid there and the graph equations are equivalent. Allowing a zero factor can introduce extra solutions, so equivalence cannot be concluded from exactness alone.

Question 8

On the domain $xy \neq 0$, consider

$$\left(\frac{1}{x} + y\right) dx + \left(\frac{1}{y} + x\right) dy = 0, \quad y(1) = 2.$$

Tasks

1. Find a potential valid on this domain, using absolute values wherever required by logarithmic integration.
2. Use the initial condition to identify the correct level. On the positive quadrant, prove that the implicit relation reduces to an explicit solution.
3. Find its maximal interval containing 1. Explain why the branch of the same algebraic hyperbola in the negative quadrant is not an extension of this IVP.
4. Analyze the different initial condition $y(1) = -1$. Determine its selected level and solution on $x > 0$, even though both coefficients vanish along that solution.

Question 8 – Solution

Strategy. Combine the logarithms using the product $u = xy$, but keep the connected domain and the sign of u in view.

Step 1: Construct the potential. Both cross partials are 1. Integration gives

$$F = \ln |x| + \ln |y| + xy = \ln |xy| + xy.$$

Indeed $F_x = 1/x + y$ and $F_y = 1/y + x$ wherever $xy \neq 0$.

Step 2: Select the positive branch. The data $(1, 2)$ give $F = \ln 2 + 2$. For $u > 0$, the function $\phi(u) = \ln u + u$ is strictly increasing, since $\phi'(u) = 1/u + 1 > 0$. Thus $\phi(u) = \phi(2)$ forces $u = 2$. Consequently

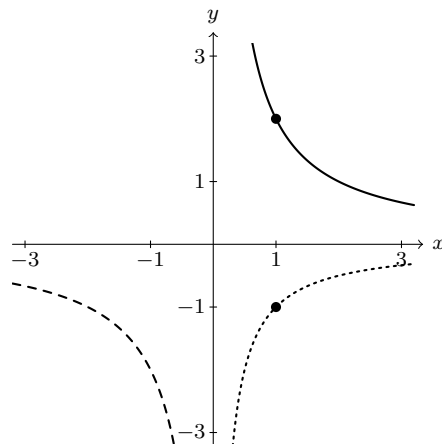
$$y = 2/x, \quad I = (0, \infty).$$

Substitution gives $M = 3/x$, $N = 3x/2$ and $y' = -2/x^2$, so $M + Ny' = 0$. The branch cannot cross $x = 0$: the equation is undefined there and $y \rightarrow +\infty$ from the right. The negative-quadrant branch $y = 2/x$, $x < 0$, is disconnected from the initial point.

Step 3: Analyze the degenerate negative level. For $(1, -1)$, $F = -1$. On $u < 0$, $\phi(u) = \ln(-u) + u$ has derivative $1/u + 1$, positive for $u < -1$ and negative for $-1 < u < 0$. Its unique maximum is $\phi(-1) = -1$. Therefore this level forces $xy = -1$, giving

$$y = -1/x \quad (x > 0).$$

Along this graph both M and N are zero, so the original equation is satisfied directly. Every solution through the point must conserve F and remain in the same quadrant, which proves this branch is unique there despite the vanishing coefficients.



Solid: $y(1) = 2$. Dashed: its disconnected hyperbola branch. Dotted: $y(1) = -1$.

Question 9

Consider

$$e^x dx + (1 + y^2) dy = 0, \quad y(0) = 0.$$

An approximate calculation claims that the selected solution passes through $(\ln 2, -1)$.

Tasks

1. Find the exact implicit solution and prove that it defines a unique differentiable graph for all real x .
2. Test the claimed point using the potential, without explicitly solving a cubic.
3. Let y_* be the true value at $x = \ln 2$. Prove $-1 < y_* < 0$ and decide whether the claimed value is too high or too low.
4. Use the mean value theorem to prove the quantitative error bound

$$\frac{1}{6} \leq |y_* - (-1)| \leq \frac{1}{3}.$$

Explain why a potential residual can provide more information than simply declaring an approximate point incorrect.

Question 9 – Solution

Strategy. Use the conserved potential to test the point, then convert its residual into an error bound using a derivative estimate.

Step 1: Find the exact relation. The cross partials are both zero. A potential is $F = e^x + y + y^3/3$, and $F(0, 0) = 1$, so

$$e^x + y + \frac{y^3}{3} = 1.$$

The function $H(u) = u + u^3/3$ is strictly increasing, onto $\mathbb{R}$, and has $H'(u) = 1 + u^2 > 0$. Thus $H(y) = 1 - e^x$ gives exactly one differentiable value for every x . Differentiation recovers the original graph equation.

Step 2: Compute the residual and locate the true value. At the claimed point, $F = 2 - 1 - 1/3 = 2/3$, so the potential residual $F - 1$ is $-1/3$. The point is not on the selected level. At $x = \ln 2$, the true value satisfies $H(y_*) = -1$. Since $H(-1) = -4/3 < -1 < H(0) = 0$, strict monotonicity gives $-1 < y_* < 0$. The claimed value -1 is too low.

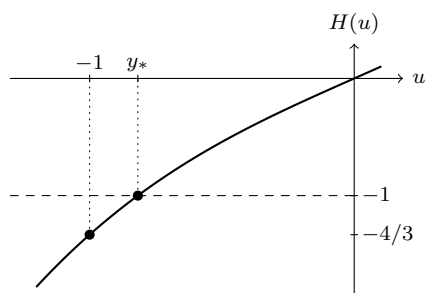
Step 3: Bound the actual error. The mean value theorem gives, for some $\xi \in (-1, y_*)$,

$$\frac{1}{3} = H(y_*) - H(-1) = H'(\xi)(y_* + 1).$$

On $[-1, 0]$, $1 \leq H' \leq 2$, so

$$\frac{1}{6} \leq y_* + 1 = |y_* - (-1)| \leq \frac{1}{3}.$$

The residual, together with monotonicity and derivative bounds, certifies both the direction and the size range of the error; no cubic formula is needed.



The true value y_* lies to the right of -1 because H is strictly increasing.

Question 10

For the exact form

$$\omega = (2xy + e^x) dx + (x^2 + 2y) dy,$$

compare two routes from $A = (0, 0)$ to $B = (1, 1)$. Route 1 goes horizontally to $(1, 0)$ and then vertically to B ; Route 2 goes vertically to $(0, 1)$ and then horizontally to B .

For an axis-aligned segment, define its accumulated value as the integral of the coefficient of dx along a horizontal segment, or of the coefficient of dy along a vertical segment, with the stated orientation.

Tasks

1. Construct a potential F normalized by $F(0, 0) = 0$.
2. Compute both route totals directly as sums of ordinary single-variable integrals.
3. Explain their equality using the chain rule and find the total around the closed route that follows Route 1 forward and Route 2 backward.
4. For the differential equation $\omega = 0$, decide whether one solution curve could pass through both A and B . Distinguish this question from the equality of the two route totals.

Question 10 – Solution

Strategy. A potential measures accumulated change along a route; a solution of the equation $\omega = 0$ must instead keep that potential constant.

Step 1: Construct a normalized potential. The cross partials are $2x$. Integrating M gives $F = x^2y + e^x + h(y)$; matching N requires $h' = 2y$. Normalization yields

$$\boxed{F = x^2y + e^x + y^2 - 1}, \quad F(B) = e + 1.$$

Step 2: Compute the routes directly. On Route 1, the horizontal and vertical contributions are

$$\int_0^1 e^x dx = e - 1, \quad \int_0^1 (1 + 2y) dy = 2,$$

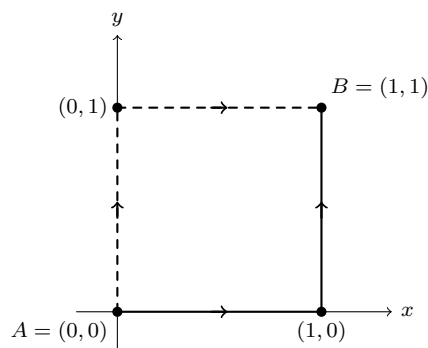
so its total is $e + 1$. On Route 2 they are

$$\int_0^1 2y dy = 1, \quad \int_0^1 (2x + e^x) dx = e,$$

again totaling $\boxed{e + 1}$.

Step 3: Explain equality and the closed route. Along a differentiable parametrized segment, $dF/dt = F_x x'(t) + F_y y'(t) = Mx' + Ny'$. Integration gives the endpoint difference of F ; these differences telescope across joined segments. Both routes therefore give $F(B) - F(A) = e + 1$. Reversing a route negates its integrals, so the closed-route total is $\boxed{0}$.

Step 4: Distinguish routes from solutions. A solution curve of $\omega = 0$ has $dF/dt = 0$ and stays on one level. Since $F(A) = 0$ and $F(B) = e + 1 \neq 0$, no such curve can pass through both points. Equal route totals express endpoint dependence; they do not make either route a solution of the differential equation.



Solid: Route 1. Dashed: Route 2. Arrows indicate travel from A toward B .

Question 1

A Bernoulli equation has the form $y' + p(x)y = q(x)y^n$. Consider the positive initial-value problem

$$y' + \frac{2}{x}y = x^3y^2, \quad y(1) = 1, \quad x > 0.$$

Tasks

1. For positive y and $n \neq 1$, derive the linear equation satisfied by $v = y^{1-n}$ using the chain rule.
2. Apply that reduction to the given IVP and solve the resulting linear equation with an integrating factor.
3. Find the maximal open interval containing 1, and determine the minimum of the selected solution on that interval.
4. Verify the answer in the original equation and identify a solution omitted by the substitution.

Question 1 – Solution

Strategy. Derive the reduction first, then retain only the interval where its inverse gives the selected positive solution.

Step 1: Reduce to a linear equation. Multiplying by y^{-n} and using $v' = (1-n)y^{-n}y'$ gives

$$v' + (1-n)pv = (1-n)q.$$

Here $n = 2$, so $v = 1/y$ obeys $v' - 2v/x = -x^3$. The integrating factor is x^{-2} , hence

$$(x^{-2}v)' = -x, \quad v = Cx^2 - \frac{x^4}{2}.$$

Since $v(1) = 1$, $C = 3/2$.

Step 2: Invert and analyze. Thus

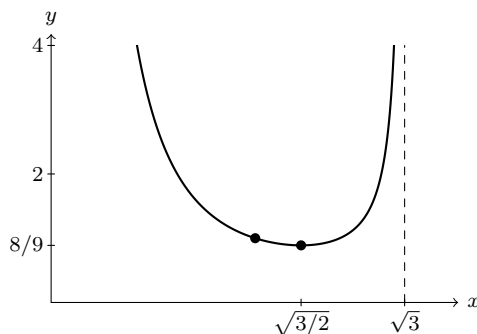
$$y = \frac{2}{x^2(3-x^2)}, \quad I = (0, \sqrt{3}).$$

The denominator $3x^2 - x^4$ is positive on I , vanishes at its endpoints, and has its unique maximum $9/4$ at $x = \sqrt{3/2}$. The solution therefore has minimum $8/9$ there and tends to $+\infty$ at both ends of I .

Step 3: Verify and restore the lost solution. With $D = 3x^2 - x^4$, $y' = -2(6x - 4x^3)/D^2$, so

$$y' + 2y/x = 4x^3/D^2 = x^3y^2, \quad y(1) = 1.$$

Also $y \equiv 0$ satisfies the original equation on $x > 0$ but is excluded by $v = 1/y$; it does not satisfy this IVP.



The selected branch has minimum $8/9$ and diverges at both interval endpoints.

Question 2

For the IVP

$$y' + y = e^{2x}y^3, \quad y(0) = -1,$$

a student sets $v = y^{-2}$ but writes $v' + 2v = 2e^{2x}$ and then plans to use $y = 1/\sqrt{v}$.

Tasks

1. Find both errors in this procedure and derive the correct equation for v .
2. Solve the linear equation and select the original solution using the initial condition.
3. Find its maximal interval containing 0, explaining why a positive transformed variable does not determine the sign of y .
4. Verify the original IVP, and state whether the opposite sign and the zero function solve the same differential equation.

Question 2 – Solution

Strategy. The derivative of y^{-2} carries a negative factor, and the inverse transformation needs a separately chosen sign.

Step 1: Correct the transformed equation. For $y \neq 0$, divide by y^3 and use $v' = -2y^{-3}y'$. This gives

$$-\frac{1}{2}v' + v = e^{2x}, \quad \boxed{v' - 2v = -2e^{2x}}.$$

The student's two coefficient signs are reversed. Moreover, $v = 1/y^2$ is the same for y and $-y$, so selecting the positive inverse would contradict the negative initial value.

Step 2: Solve and choose the branch. The integrating factor is e^{-2x} :

$$(e^{-2x}v)' = -2, \quad v = e^{2x}(C - 2x).$$

The condition $v(0) = 1$ gives $C = 1$. Continuity and $y(0) = -1$ select

$$\boxed{y = -\frac{e^{-x}}{\sqrt{1-2x}}, \quad I = (-\infty, 1/2)}.$$

Here $v > 0$. At $x \uparrow 1/2$, the solution tends to $-\infty$, so there is no finite extension; for $x > 1/2$, this v cannot equal $1/y^2$ for real y .

Step 3: Verify and distinguish other solutions. On I , logarithmic differentiation of $|y|$ gives

$$\frac{y'}{y} = -1 + \frac{1}{1-2x}.$$

Consequently $y' + y = y/(1-2x) = e^{2x}y^3$, and the initial value is -1 . The opposite branch $+e^{-x}/\sqrt{1-2x}$ also solves the differential equation, with initial value 1 . The zero function solves it as well but is omitted by y^{-2} . Neither of those alternatives solves the stated IVP.

Question 3

Seek nonnegative, continuously differentiable solutions for $t \geq 0$ of

$$y' + 2y = 2\sqrt{y}, \quad y(0) = 0,$$

where the prime denotes d/dt and $\sqrt{y}$ is the nonnegative square root.

Tasks

1. On an interval where $y > 0$, derive and solve the equation for $w = \sqrt{y}$.
2. Find every solution that waits at zero until a time $b \geq 0$ and then becomes positive. Include the possibility of staying zero forever.
3. Check differentiability and the original equation at the joining time, and justify that your list is exhaustive for $t \geq 0$.
4. Explain why squaring a linear formula with $w < 0$ does not give a valid solution of the original equation on that interval.

Question 3 – Solution

Strategy. The linearized equation applies only on positive pieces; restore zero intervals and enforce the sign of the square root.

Step 1: Solve a positive piece. Writing $y = w^2$ with $w > 0$ gives $2ww' + 2w^2 = 2w$, hence $w' + w = 1$. Thus $w = 1 + Ce^{-t}$. If positivity starts at b with $w(b) = 0$, then $w = 1 - e^{-(t-b)}$, positive precisely for $t > b$.

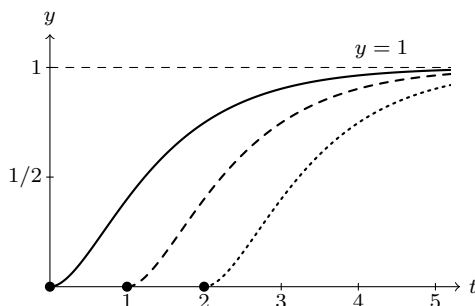
Step 2: Restore the zero intervals. For each finite $b \geq 0$, define

$$y_b(t) = \begin{cases} 0, & 0 \leq t \leq b, \\ (1 - e^{-(t-b)})^2, & t \geq b. \end{cases}$$

Also include $y \equiv 0$, denoted by $b = \infty$. On the positive piece let $r = e^{-(t-b)}$; then $y' = 2r(1 - r)$ and $y' + 2y = 2(1 - r) = 2\sqrt{y}$. At a positive joining time, both one-sided derivatives are 0; if $b = 0$, the right derivative is 0. Thus the solution is C^1 on its stated domain and satisfies the equation at the join.

Step 3: Prove completeness and reject a false extension. Any positive component must start from zero at its finite left endpoint, including $b = 0$, and the linear formula above fixes that component uniquely. It stays positive forever after b , so it cannot end at another zero. Hence there is at most one positive component and the list is exhaustive.

For $t < b$, the same linear expression has $w < 0$. If it is squared, $\sqrt{w^2} = |w| = -w$, while the computed left side is $2w$. Thus it fails the original equation there. Squaring cannot remove this sign restriction.



Solid: $b = 0$. Dashed: $b = 1$. Dotted: $b = 2$. Each is zero before its departure.

Question 4

Consider the negative-power Bernoulli equation

$$y' + y = \frac{e^{-x}}{y}, \quad y(0) = -1.$$

The original equation is defined only when $y \neq 0$.

Tasks

1. Derive the linear equation for $v = y^2$ and solve it using the initial condition.
2. Select the real branch for y and find its maximal interval containing 0.
3. Determine what happens to y and y' at its finite endpoint. Decide whether reaching the value zero permits continuation.
4. Verify the equation, and explain why the zero function cannot be restored as an omitted solution in this example.

Question 4 – Solution

Strategy. The squared variable must be strictly positive, and the original reciprocal term remains undefined at zero.

Step 1: Linearize and integrate. Multiplication by $2y$ yields $v' + 2v = 2e^{-x}$, where $v = y^2$. Using the integrating factor e^{2x} ,

$$(e^{2x}v)' = 2e^x, \quad v = 2e^{-x} + Ce^{-2x}.$$

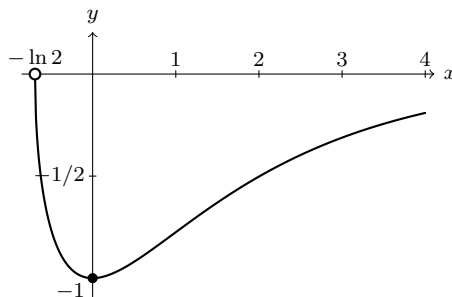
Since $v(0) = 1$, $C = -1$.

Step 2: Invert on the correct interval. The condition $v = e^{-2x}(2e^x - 1) > 0$ requires $x > -\ln 2$. The initial sign gives

$$y = -e^{-x}\sqrt{2e^x - 1}, \quad I = (-\ln 2, \infty).$$

As $x \downarrow -\ln 2$, $y \rightarrow 0$ from below. The equation gives $y' = e^{-x}/y - y \rightarrow -\infty$, so no differentiable extension exists at that endpoint. Also the original right-hand side is undefined there. As $x \rightarrow \infty$, $y \rightarrow 0$ from below, but it never reaches zero at a finite point of I .

Step 3: Verify and check the excluded value. Differentiating $y^2 = v$ gives $2yy' = v'$. Since $v' + 2v = 2e^{-x}$ and $y \neq 0$ on I , division by $2y$ recovers $y' + y = e^{-x}/y$. The selected formula gives $y(0) = -1$. Unlike positive-power examples, $y \equiv 0$ is not a solution: the original equation has no value at zero.



The finite endpoint is excluded; the negative branch approaches zero again as $x \rightarrow \infty$.

Question 5

For each real initial value a , consider forward solutions of

$$y' + y = y^3, \quad y(0) = a, \quad t \geq 0,$$

where the prime denotes d/dt .

Tasks

1. Solve using $v = y^{-2}$ when $a \neq 0$, retaining the sign of a . Treat $a = 0$ separately.
2. Classify all a according to whether the forward solution tends to zero, is constant, or blows up in finite time.
3. In the blow-up cases, find the exact lifetime and the sign of the divergence.
4. Verify the solution and explain why the threshold initial values are not included in either neighboring regime.

Question 5 – Solution

Strategy. A linear denominator exposes the threshold, while the sign of the original solution must be carried separately.

Step 1: Solve the transformed IVP. For $a \neq 0$, $v = y^{-2}$ gives $v' - 2v = -2$, so

$$v = 1 + (a^{-2} - 1)e^{2t}, \quad y = \frac{a}{\sqrt{a^2 + (1 - a^2)e^{2t}}}.$$

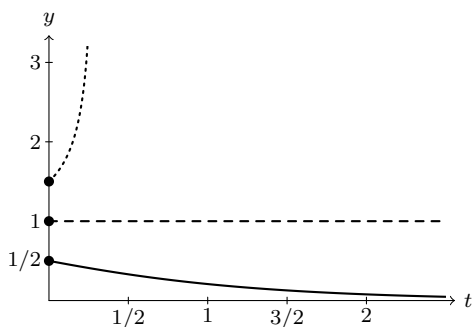
The denominator is the positive square root on the interval through 0. For $a = 0$, the unique solution is $y \equiv 0$; the smooth right-hand side $y^3 - y$ ensures local uniqueness.

Step 2: Classify the thresholds. If $|a| < 1$, the denominator is positive for every $t \geq 0$ and $y \rightarrow 0$. This includes $a = 0$. If $a = \pm 1$, the formula is constant: $y \equiv \pm 1$. If $|a| > 1$, the denominator first vanishes at

$$T = \frac{1}{2} \ln \frac{a^2}{a^2 - 1}.$$

The maximal forward lifetime is $0 \leq t < T$. As $t \uparrow T$, $y \rightarrow +\infty$ for $a > 1$ and $y \rightarrow -\infty$ for $a < -1$.

Step 3: Verify and interpret. For nonzero branches, $v' = -2y^{-3}y' = 2v - 2$ implies $y' = -y + y^3$, and the formula gives $y(0) = a$. The three constants 0, 1, -1 also check directly. At $a = \pm 1$ the exponential term in v is exactly zero; these solutions neither decay nor blow up. A threshold case must be substituted into the equation, not inferred from neighboring values.



Solid: $a = 1/2$. Dashed: $a = 1$. Dotted: $a = 3/2$. Negative solutions are reflections.

Question 6

On $x > 0$, two positive solutions of one unknown Bernoulli equation

$$y' + p(x)y = q(x)y^2$$

are known exactly:

$$y_1(x) = \frac{1}{1+x}, \quad y_2(x) = \frac{1}{1+2x}.$$

Assume p and q are continuous there.

Tasks

1. Transform both solutions using $v = 1/y$ and recover $p(x)$ and $q(x)$ uniquely.
2. Verify both supplied solutions directly in the recovered nonlinear equation.
3. Solve that equation with the different condition $y(1) = 2$, and find its maximal interval within $x > 0$.
4. Explain why subtracting the original nonlinear solutions is less useful here than subtracting their transformed equations, and identify the solution lost by taking reciprocals.

Question 6 – Solution

Strategy. The reciprocal transformation turns coefficient recovery into two linear identities; their difference eliminates the forcing term.

Step 1: Recover the coefficients. Each transformed solution satisfies $v' - pv = -q$. Here $v_1 = 1 + x$ and $v_2 = 1 + 2x$, so

$$1 - p(1 + x) = -q, \quad 2 - p(1 + 2x) = -q.$$

Subtracting gives $1 - px = 0$. Since $x > 0$, this fixes $p = 1/x$ uniquely. Substituting back gives $q = 1/x$. Thus

$$\boxed{y' + \frac{y}{x} = \frac{y^2}{x}}.$$

Step 2: Verify the given data. For $y = 1/(1 + kx)$, with $k = 1$ or 2 ,

$$y' + \frac{y}{x} = -\frac{k}{(1 + kx)^2} + \frac{1}{x(1 + kx)} = \frac{1}{x(1 + kx)^2} = \frac{y^2}{x}.$$

Both functions are positive and defined throughout $x > 0$.

Step 3: Construct the new solution. The transformed equation is $v' - v/x = -1/x$. Multiplying by $1/x$ gives $(v/x)' = -1/x^2$, hence $v = 1 + Cx$. Since $y(1) = 2$, $v(1) = 1/2$, so $C = -1/2$. Therefore

$$\boxed{y = \frac{1}{1 - x/2}, \quad I = (0, 2)}.$$

This is the maximal interval within the specified domain through 1: the solution has a pole at 2, and the coefficients are undefined at 0.

Step 4: Explain the advantage and restore zero. Subtracting the transformed equations gives the homogeneous linear relation $\Delta v' - p\Delta v = 0$. Subtracting the original equations leaves $q(y_2^2 - y_1^2)$ and does not give that linear elimination. The function $\boxed{y \equiv 0}$ satisfies the recovered equation but is excluded by the reciprocal transformation.

Question 7

Suppose y_1, y_2 are positive solutions on an interval of

$$y' + p(x)y = q(x)y^3.$$

Let $0 < \alpha < 1$.

Tasks

1. Prove that $v = \alpha y_1^{-2} + (1 - \alpha)y_2^{-2}$ solves the transformed linear equation and gives a positive solution $y = v^{-1/2}$.
2. Show that this new solution lies between y_1 and y_2 pointwise.
3. Apply the construction with $\alpha = 1/4$ to the equation $y' + xy = xy^3$ and the solutions $y_1 = 1$, $y_2 = (1 + e^{x^2})^{-1/2}$. Verify the supplied solutions through their transforms.
4. Prove that the ordinary arithmetic mean $m = (y_1 + y_2)/2$ of these two supplied solutions fails the nonlinear equation at $x = 1$.

Question 7 – Solution

Strategy. Affine combinations preserve the common forcing in the transformed linear equation; they do not generally preserve the original cubic equation.

Step 1: Combine the transformed solutions. Set $v_i = y_i^{-2}$. Both satisfy $v'_i - 2pv_i = -2q$. Multiplying by α and $1 - \alpha$ and adding yields $v' - 2pv = -2q$. Positivity of v_1, v_2 implies $v > 0$, so $y = v^{-1/2}$ is well defined. Differentiating this inverse recovers $y' + py = qy^3$.

Step 2: Prove the pointwise bound. At each point, v lies between v_1 and v_2 . The function $u \mapsto u^{-1/2}$ is decreasing on $u > 0$, so reversing their order still puts y between y_1 and y_2 . Equality occurs when the supplied values agree; otherwise the bounds are strict.

Step 3: Apply the construction. For the given equation, the transformed equation is $v' - 2xv = -2x$. The supplied transforms are $v_1 = 1$ and $v_2 = 1 + e^{x^2}$; direct differentiation verifies both. Their weighted combination gives

$$v = 1 + \frac{3}{4}e^{x^2}, \quad \boxed{y = \frac{1}{\sqrt{1 + \frac{3}{4}e^{x^2}}}} \quad (x \in \mathbb{R}).$$

It is positive and lies strictly between the two supplied solutions.

Step 4: Reject ordinary superposition. Because the original functions solve the equation,

$$m' + xm = \frac{x}{2}(y_1^3 + y_2^3).$$

At $x = 1$, put $a = y_1(1)$ and $b = y_2(1)$. They are distinct and positive. The residual is

$$m' + m - m^3 = \frac{a^3 + b^3}{2} - \frac{(a + b)^3}{8} = \frac{3}{8}(a + b)(a - b)^2 > 0.$$

Thus the arithmetic mean fails the nonlinear equation. The valid averaging occurs after the Bernoulli transformation.

Question 8

Consider

$$y' + xy = e^{x^2} y^2, \quad y(0) = a, \quad a \in \mathbb{R}.$$

Define $J(x) = \int_0^x e^{t^2/2} dt$; an elementary antiderivative is not required.

Tasks

1. Solve the IVP for $a \neq 0$ by the Bernoulli substitution, expressing the answer in terms of J . Treat $a = 0$ separately.
2. Prove that J is strictly increasing onto $\mathbb{R}$ and that the equation $J(T) = 1/a$ has exactly one real solution.
3. Determine the maximal interval containing 0 for each sign of a , and the sign of the divergence at its finite endpoint.
4. Verify the formula using the fundamental theorem of calculus and explain why an exact definite-integral expression is a complete answer.

Question 8 – Solution

Strategy. Keep the forcing integral exact, then use its monotonicity to determine the entire admissible denominator interval.

Step 1: Linearize and integrate. For $a \neq 0$, set $v = 1/y$. Then $v' - xv = -e^{x^2}$, with integrating factor $e^{-x^2/2}$. Thus

$$(e^{-x^2/2}v)' = -e^{x^2/2}, \quad v = e^{x^2/2} \left(\frac{1}{a} - J(x) \right).$$

Inverting gives

$$y(x) = \frac{ae^{-x^2/2}}{1 - aJ(x)}.$$

For $a = 0$, $y \equiv 0$ is the unique solution, since the original right-hand side is smooth in y .

Step 2: Locate the unique obstruction. By the fundamental theorem, $J' = e^{x^2/2} > 0$. The integrand is even, so J is odd. Also $J(x) \geq x$ for $x \geq 0$, so its limits at the two infinities are $\pm\infty$. Therefore it is one-to-one and onto, and there is exactly one T with $J(T) = 1/a$.

Step 3: Select the interval and sign. If $a > 0$, then $T > 0$, and the maximal interval is $(-\infty, T)$. The denominator approaches zero from above as $x \uparrow T$, so $y \rightarrow +\infty$. If $a < 0$, then $T < 0$, and the maximal interval is (T, ∞) ; as $x \downarrow T$, the denominator approaches zero from above and $y \rightarrow -\infty$. A pole cannot be crossed by a finite differentiable continuation.

Step 4: Verify and interpret. With $D = 1 - aJ$, differentiation using $J' = e^{x^2/2}$ gives

$$y' = -xy + \frac{a^2}{D^2} = -xy + e^{x^2} y^2.$$

Also $J(0) = 0$ gives $y(0) = a$. The definite integral specifies an exact differentiable function and uniquely determines the endpoint; elementary notation is unnecessary for either the solution or its interval.

Question 9

Seek strictly positive, 2π -periodic solutions, defined for every real x , of

$$y' + y = (2 + \sin x)y^2.$$

Tasks

1. Use the reciprocal substitution to find the general transformed solution.
2. Prove that there is exactly one strictly positive 2π -periodic solution and find its initial value at $x = 0$.
3. Determine its minimum and maximum over one period.
4. Show that changing the initial value to $y(0) = 1/2$ causes a finite positive-time blow-up, even though the periodic solution remains bounded. Verify the original equation for the periodic solution.

Question 9 – Solution

Strategy. Periodicity removes the exponentially growing homogeneous term in the reciprocal variable; positivity must then be checked before inversion.

Step 1: Solve the linear equation. With $v = 1/y$, the equation becomes $v' - v = -(2 + \sin x)$. A particular solution and the general solution are

$$v_p = 2 + \frac{\sin x + \cos x}{2}, \quad v = v_p + Ce^x.$$

Indeed $v'_p - v_p = -2 - \sin x$.

Step 2: Select and bound the periodic solution. If positive y is 2π -periodic, then so is $v = 1/y$. The term Ce^x is periodic only if $C = 0$. Since

$$2 - \frac{\sqrt{2}}{2} \leq v_p \leq 2 + \frac{\sqrt{2}}{2},$$

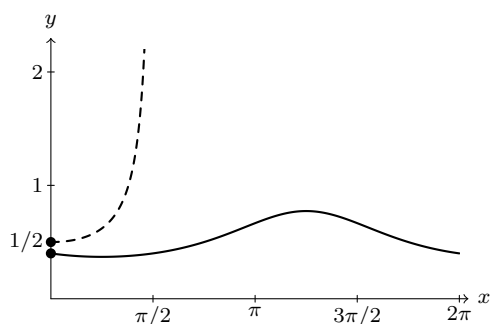
its reciprocal exists globally and is positive. Therefore

$$y_p = \frac{1}{2 + (\sin x + \cos x)/2}, \quad y_p(0) = \frac{2}{5}.$$

Its minimum is $1/(2 + \sqrt{2}/2)$ at $x = \pi/4$, and its maximum is $1/(2 - \sqrt{2}/2)$ at $x = 5\pi/4$, modulo 2π .

Step 3: Check the perturbed initial condition. For $y(0) = 1/2$, $v(0) = 2$, so $C = -1/2$. The function $v = v_p - e^x/2$ starts positive, but $v(2\pi) = 5/2 - e^{2\pi}/2 < 0$. It therefore has a first zero $T \in (0, 2\pi)$. At this zero, $v'(T) = -(2 + \sin T) < 0$. Hence $v \downarrow 0$ from the positive side and $y = 1/v \rightarrow +\infty$ as $x \uparrow T$.

Step 4: Verify the periodic branch. Since $v'_p = v_p - (2 + \sin x)$, differentiation gives $y'_p = -y_p + (2 + \sin x)y_p^2$, as required.



Solid: the positive periodic solution. Dashed: $y(0) = 1/2$, which blows up before a full cycle.

Question 10

Consider positive solutions, near $x = 0$, of the exponent-dependent IVP

$$y' + y = y^n, \quad y(0) = 2, \quad n \in \mathbb{R}.$$

For $y > 0$, interpret $y^n = \exp(n \ln y)$.

Tasks

1. Solve the cases $n = 0$ and $n = 1$ directly. Explain why they do not require the usual nonlinear Bernoulli reduction.
2. For $n \neq 1$, derive the solution formula and state the positivity condition needed to interpret its real power.
3. Check that the general formula agrees with the direct answer at $n = 0$ and verifies the initial condition for every admissible n .
4. For each fixed real x , prove that the formula tends to the $n = 1$ solution as $n \rightarrow 1$. Do not substitute $n = 1$ into an expression with exponent $1/(1 - n)$.

Question 10 – Solution

Strategy. Handle the exceptional equations directly, then analyze the apparent singularity of the general expression through its logarithm.

Step 1: Solve the elementary exceptional cases. If $n = 0$, the equation is $y' + y = 1$, giving $y = 1 + e^{-x}$. If $n = 1$, the two y terms cancel, giving $y = 2$. Both are positive on $\mathbb{R}$; the first is linear and the second has zero derivative.

Step 2: Derive the parameterized formula. For $n \neq 1$, set $h = 1 - n \neq 0$ and $v = y^h$. Then

$$v' + hv = h, \quad v(0) = 2^h,$$

so $v = 1 + (2^h - 1)e^{-hx}$. Therefore

$$y_n(x) = \left[1 + (2^{1-n} - 1)e^{-(1-n)x} \right]^{1/(1-n)}.$$

Use the connected interval through 0 on which the bracket is strictly positive. This is necessary because the original positive solution has $v = y^h > 0$; a negative base cannot be accepted by choosing a special rational exponent.

Step 3: Check the consistency conditions. At $n = 0$, the formula becomes $1 + e^{-x}$. At $x = 0$, its bracket is 2^{1-n} , and the positive-real power gives 2. Conversely, differentiating $v = y^h$ on the stated interval recovers the original equation, so the inversion is valid there.

Step 4: Take the limit without dividing by zero. For fixed x , let $B(h) = 1 + (e^{h \ln 2} - 1)e^{-hx}$. Since $B(0) = 1$, it is positive for all sufficiently small $|h|$. Now

$$\ln y_n = \frac{\ln B(h)}{h}, \quad B'(0) = \ln 2.$$

The derivative definition gives $\lim_{h \rightarrow 0} \ln B(h)/h = \ln 2$. Exponentiating yields $\lim_{n \rightarrow 1} y_n(x) = 2$, exactly the direct $n = 1$ solution. Joint continuity on the compact segment between 0 and the fixed x keeps the bracket positive there for small $|h|$, so these values belong to the IVP branch.

Question 1

Consider the homogeneous first-order equation

$$y' = \frac{y}{x} + \sqrt{1 + (y/x)^2}, \quad y(1) = 0, \quad x > 0.$$

Here “homogeneous” means that the right-hand side depends only on y/x ; it does not mean a homogeneous linear equation.

Tasks

1. Set $v = y/x$ and derive the equation for v , showing the product-rule term in y' .
2. Solve the transformed IVP and recover an explicit formula for y .
3. Verify the original equation, taking care with the nonnegative square root, and find the maximal interval within $x > 0$.
4. A student uses $y' = xv'$ when $y = xv$. Identify the omitted term and explain why that error changes the transformed equation.

Question 1 – Solution

Strategy. The ratio substitution removes the repeated y/x , but differentiating $y = xv(x)$ requires both product-rule terms.

Step 1: Transform the equation. Since $y' = v + xv'$, substitution gives

$$v + xv' = v + \sqrt{1 + v^2}, \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

The initial value is $v(1) = 0$. An antiderivative on the left is $\ln(v + \sqrt{1 + v^2})$, whose logarithm has a strictly positive argument for every real v . Hence

$$\ln(v + \sqrt{1 + v^2}) = \ln x.$$

Step 2: Recover the original unknown. Exponentiating gives $v + \sqrt{1 + v^2} = x$. Squaring $\sqrt{1 + v^2} = x - v$ and simplifying gives

$$v = \frac{x^2 - 1}{2x}, \quad \boxed{y = \frac{x^2 - 1}{2}}.$$

This step is valid: for the recovered expression, $x - v = (x^2 + 1)/(2x) > 0$ on $x > 0$, so the unsquared relation also holds.

Step 3: Verify the branch and interval. For this y ,

$$\sqrt{1 + (y/x)^2} = \frac{x^2 + 1}{2x}, \quad \frac{y}{x} + \sqrt{1 + (y/x)^2} = x = y'.$$

Also $y(1) = 0$. The formula satisfies the original equation for every $x > 0$, so its maximal interval there is $\boxed{(0, \infty)}$. The polynomial has a value at 0, but the given equation does not; an algebraic extension alone does not extend this solution through the excluded point.

Step 4: Diagnose the missing derivative term. The omitted term is v . Writing only xv' would produce $xv' = v + \sqrt{1 + v^2}$ instead of $xv' = \sqrt{1 + v^2}$. The two equations describe different ratio functions, so the product rule is essential to the reduction.

Question 2

On $x > 0$, consider

$$y' = \frac{y}{x} + (y/x - 1)(y/x - 2), \quad y(1) = 3.$$

Tasks

1. Use $v = y/x$ to derive a separable equation. Find every constant value of v before dividing by a factor involving v .
2. Solve the transformed equation, and give the resulting original solution family together with any missing straight-line solution.
3. Select the IVP solution and its maximal interval containing 1.
4. Verify the IVP solution and explain why it cannot cross either straight-line solution within that interval.

Question 2 – Solution

Strategy. Constant ratios correspond to straight lines, so preserve them before separating the nonconstant ratio equation.

Step 1: Transform and preserve the constants. Writing $y = xv$ gives $xv' = (v-1)(v-2)$. The constant ratios $v = 1, 2$ yield the solutions $y = x, y = 2x$. For other ratios,

$$\int \frac{dv}{(v-1)(v-2)} = \int \frac{dx}{x}, \quad \ln \left| \frac{v-2}{v-1} \right| = \ln x + C_0.$$

Thus $(v-2)/(v-1) = Cx$, with nonzero C for nonconstant v , and

$$y = \frac{x(2 - Cx)}{1 - Cx}.$$

Allowing $C = 0$ restores $y = 2x$; $y = x$ remains a separate solution. Each formula is used only on intervals where its denominator is nonzero.

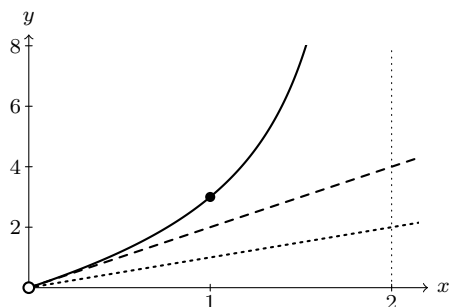
Step 2: Select the IVP branch. The condition $v(1) = 3$ gives $C = 1/2$, hence

$$y = \frac{x(2 - x/2)}{1 - x/2}, \quad I = (0, 2).$$

The coefficient domain excludes 0, while $y \rightarrow +\infty$ as $x \uparrow 2$.

Step 3: Verify and compare. For $v = (2 - Cx)/(1 - Cx)$, $v' = C/(1 - Cx)^2$, and $(v-1)(v-2) = Cx/(1 - Cx)^2 = xv'$. Thus $y' = v + xv'$ recovers the original equation, and the selected initial value is 3.

On $0 < x < 2$, $v = 1 + 1/(1 - x/2) > 2$. Therefore $y > 2x > x$ throughout the interval: neither line is crossed.



Solid: IVP solution. Dashed: $y = 2x$. Dotted sloping line: $y = x$. The domain is $x > 0$.

Question 3

Consider an IVP on the negative half-line:

$$y' = \frac{y}{x} + \frac{x}{y}, \quad y(-1) = 1, \quad x < 0,$$

with the original equation defined only when $y \neq 0$.

Tasks

1. Use $v = y/x$ and separate the equation, keeping the correct logarithmic absolute value.
2. Select both the constant and the sign of the transformed solution from the initial condition.
3. Recover y and determine its maximal interval containing -1 within $x < 0$.
4. Verify the equation and describe the finite endpoint. Explain why replacing $\ln|x|$ by $\ln x$, or choosing the positive sign for v , would fail here.

Question 3 – Solution

Strategy. The substitution is valid for negative x too, but its initial ratio is negative and the logarithm must respect the domain.

Step 1: Transform and integrate. From $y = xv$ and $y' = v + xv'$,

$$xv' = \frac{1}{v}, \quad v \, dv = \frac{dx}{x}, \quad v^2 = 2 \ln |x| + C.$$

At $x = -1$, $v = y/x = -1$, so $C = 1$. Continuity and the nonzero ratio select

$$v = -\sqrt{1 + 2 \ln |x|}.$$

Step 2: Recover the branch and domain. The radicand must be strictly positive, since zero would give the excluded value $y = 0$. Thus $|x| > e^{-1/2}$. On the negative half-line this gives

$$\boxed{y = -x\sqrt{1 + 2 \ln |x|}, \quad I = (-\infty, -e^{-1/2})}.$$

It is positive and gives $y(-1) = 1$.

Step 3: Verify and check the endpoint. Let $D = 1 + 2 \ln |x|$. Since $D' = 2/x$,

$$y' = -\sqrt{D} - \frac{1}{\sqrt{D}}, \quad \frac{y}{x} + \frac{x}{y} = -\sqrt{D} - \frac{1}{\sqrt{D}}.$$

At the finite endpoint approached from the left, $D \downarrow 0$, so $y \rightarrow 0$ from above and $y' \rightarrow -\infty$. The original equation is undefined there, and no differentiable extension through it is possible.

Step 4: Identify the two domain mistakes. The real logarithm $\ln x$ is not defined for these negative x ; the antiderivative is $\ln |x|$. Also $v > 0$ would give $y = xv < 0$ on this half-line, contradicting $y(-1) = 1$. The sign must be selected for the transformed ratio, not guessed from the sign of y alone.

Question 4

Consider the nonseparable-looking equation

$$y' = (x + y)^2 - 1, \quad y(0) = 1.$$

Tasks

1. Set $u = x + y$. Derive its differential equation, including the derivative of x .
2. Solve the transformed IVP, recover y , and find its maximal interval containing 0.
3. Determine the minimum of the selected solution and verify the original equation.
4. Find the original solution represented by the constant transformed value $u = 0$. Explain why dividing by u^2 would miss it and why it is not a constant function y .

Question 4 – Solution

Strategy. Replace the repeated linear combination by one variable. The extra derivative of x cancels the constant term in the equation.

Step 1: Reduce to a separable IVP. For $u = x + y$, $u' = 1 + y' = u^2$, and $u(0) = 1$. Separating on the nonzero branch gives

$$-\frac{1}{u} = x + C, \quad C = -1, \quad u = \frac{1}{1-x}.$$

Thus

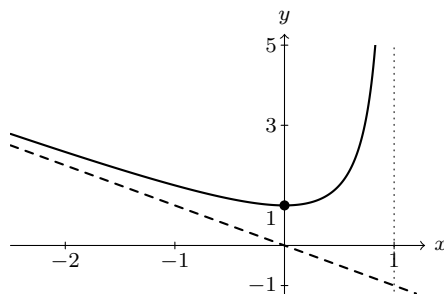
$$y = -x + \frac{1}{1-x}, \quad I = (-\infty, 1).$$

The only finite obstruction is the pole at $x = 1$, where $y \rightarrow +\infty$ from the left.

Step 2: Find the minimum and verify. Differentiation gives $y' = -1 + (1-x)^{-2}$. On I this is negative for $x < 0$, zero at 0, and positive for $0 < x < 1$. Hence the unique global minimum on I is $y(0) = 1$.

Also $x + y = 1/(1-x)$, so $(x + y)^2 - 1 = (1-x)^{-2} - 1 = y'$, and the initial value checks directly.

Step 3: Restore the constant transformed branch. The equation $u' = u^2$ also has $u \equiv 0$. In the original variables it gives $y = -x$ on all of $\mathbb{R}$. Its derivative is -1 and its right-hand side is $0^2 - 1 = -1$, so it is valid. Division by u^2 excludes it. A constant value of a combination of x and y need not represent a constant value of y .



Solid: IVP solution, with minimum $(0, 1)$. Dashed: the restored solution $y = -x$.

Question 5

Consider

$$y' = \frac{x + y - 3}{x - y + 1}, \quad y(2) = 2,$$

on the domain $x - y + 1 \neq 0$.

Tasks

1. Find constants h, k such that $X = x - h, Y = y - k$ remove the constant terms in both affine expressions. Write the shifted equation and initial point.
2. On a neighborhood of that point with $X > 0$, use $v = Y/X$ to obtain an implicit solution in X, Y .
3. Verify the local implicit relation and find the slope at the original initial point.
4. Is $X = 0$ automatically a singular line of the original differential equation? Identify the actual excluded line and explain the difference between these restrictions.

Question 5 – Solution

Strategy. Translate the intersection of the two affine lines to the origin, then apply the homogeneous ratio substitution locally.

Step 1: Choose the translation. The equations $h + k - 3 = 0$ and $h - k + 1 = 0$ give $\boxed{h = 1, k = 2}$. Hence

$$X = x - 1, \quad Y = y - 2, \quad \frac{dY}{dX} = \frac{X + Y}{X - Y}, \quad (X, Y) = (1, 0) \text{ initially.}$$

A translation does not change the derivative because $dX/dx = 1$.

Step 2: Substitute and integrate. With $Y = Xv$,

$$v + Xv' = \frac{1 + v}{1 - v}, \quad \frac{1 - v}{1 + v^2} dv = \frac{dX}{X}.$$

On the local region $X > 0$, integration gives

$$\arctan v - \frac{1}{2} \ln(1 + v^2) = \ln X + C.$$

At $(1, 0)$, $C = 0$. Substituting $v = Y/X$ and simplifying yields

$$\boxed{\arctan(Y/X) - \frac{1}{2} \ln(X^2 + Y^2) = 0}$$

for the local branch through $(1, 0)$ with $X > 0$.

Step 3: Verify and find the slope. For the displayed left side F , differentiation gives

$$F_X = -\frac{X + Y}{X^2 + Y^2}, \quad F_Y = \frac{X - Y}{X^2 + Y^2}.$$

Thus $F_X + F_Y Y' = 0$ recovers the shifted equation wherever $X - Y \neq 0$. At $(1, 0)$, $F_Y = 1$, so a local graph exists and $\boxed{y'(2) = 1}$.

Step 4: Separate coordinate and equation restrictions. The ratio Y/X excludes $X = 0$, namely $x = 1$, but the original equation is defined there whenever $y \neq 2$. Its actual excluded line is $\boxed{x - y + 1 = 0}$, or $y = x + 1$. An alternative local coordinate or angle branch can be needed when the ratio chart fails; that failure alone does not prove that an original solution ends. No global branch claim is made by this local arctangent formula.

Question 6

Consider an equation with parallel affine expressions:

$$y' = \frac{2x + y + 1}{4x + 2y + 3}, \quad y(0) = 0,$$

where $4x + 2y + 3 \neq 0$.

Tasks

1. Explain why no translation of x, y removes the constants in both numerator and denominator simultaneously.
2. Use $u = 2x + y + 1$ to derive a separable equation, then obtain an implicit formula for the IVP.
3. Find the solution corresponding to a constant u that would be lost during separation, and verify it in the original equation.
4. Prove that the selected implicit formula defines the IVP solution for every real x , while remaining inside the original coefficient domain.

Question 6 – Solution

Strategy. When the affine forms cannot be centered together, use their shared linear combination directly.

Step 1: Reject a simultaneous centering. A translation by (h, k) would require $2h + k + 1 = 0$ and $4h + 2k + 3 = 0$. Doubling the first equation gives $4h + 2k + 2 = 0$, contradicting the second. Their zero lines are parallel and distinct.

Step 2: Reduce and integrate. Set $u = 2x + y + 1$. Then the denominator is $2u + 1$, and

$$u' = 2 + y' = \frac{5u + 2}{2u + 1}, \quad \frac{2u + 1}{5u + 2} du = dx.$$

The original domain excludes $u = -1/2$. For $u \neq -2/5$, an antiderivative is

$$H(u) = \frac{2}{5}u + \frac{1}{25} \ln |5u + 2|.$$

Since $u(0) = 1$, the IVP relation is

$$H(u) = x + \frac{2}{5} + \frac{1}{25} \ln 7, \quad y = u - 2x - 1.$$

Step 3: Restore the lost constant combination. The transformed equation admits $u \equiv -2/5$, giving $y = -2x - 7/5$. Its numerator is $-2/5$ and its denominator is $1/5$, so the right side is $-2 = y'$. It is a global solution, but its initial value is not 0.

Step 4: Prove the selected branch is global. The initial value lies in $u > -2/5$. On this interval,

$$H'(u) = \frac{2u + 1}{5u + 2} > 0,$$

and $H(u)$ tends to $-\infty$ at the left endpoint and to $+\infty$ as $u \rightarrow \infty$. Thus the implicit relation gives exactly one differentiable u for every real x . Also $2u + 1 > 1/5$, so the original denominator never vanishes. Differentiating $H(u) = x + H(1)$ gives the transformed equation and then the original equation; this verifies the unique global IVP branch.

Question 7

Consider

$$y' = xe^{-y} - 1, \quad y(0) = 0.$$

Tasks

1. Set $u = e^y$ and derive a linear differential equation for u .
2. Solve it using the initial condition and recover y .
3. Prove that the selected transformed solution stays positive for every real x . Find the minimum value of y and where it occurs.
4. The same linear equation has the solution $u = x - 1$. Determine precisely where it produces a real original solution, and explain why a valid transformed solution need not be invertible everywhere.

Question 7 – Solution

Strategy. Exponentiating the unknown makes the reciprocal exponential linear, but the inverse logarithm requires a positive transformed value.

Step 1: Transform and solve. For $u = e^y$, $u' = e^y y' = x - u$. The initial value is $u(0) = 1$, so

$$u' + u = x, \quad (e^x u)' = x e^x,$$

which gives $u = x - 1 + C e^{-x}$. The data select $C = 2$, hence

$$y(x) = \ln(x - 1 + 2e^{-x}).$$

Step 2: Establish the inverse domain and minimum. Let $U(x) = x - 1 + 2e^{-x}$. Then $U' = 1 - 2e^{-x}$ and $U'' = 2e^{-x} > 0$. Its unique global minimum occurs at $x = \ln 2$ and equals $\ln 2 > 0$. Therefore the logarithm is defined and smooth for all real x , so the IVP interval is $\boxed{\mathbb{R}}$.

Since the logarithm is increasing, the solution has unique minimum

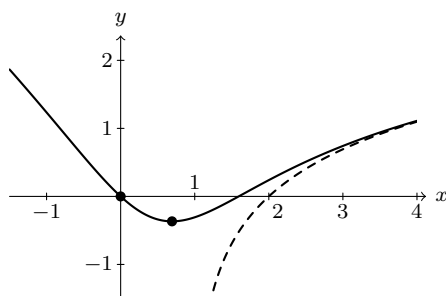
$$y(\ln 2) = \ln(\ln 2).$$

Verification is immediate from $U' = x - U$: $y' = U'/U = x/U - 1 = x e^{-y} - 1$, and $y(0) = \ln 1 = 0$.

Step 3: Test another transformed solution. The linear solution $u = x - 1$ is defined everywhere, but $u > 0$ only for $x > 1$. It gives

$$y = \ln(x - 1), \quad x > 1.$$

On that interval, $y' = 1/(x - 1) = x/(x - 1) - 1$, verifying the original equation. At $x \downarrow 1$, $y \rightarrow -\infty$, and for $x \leq 1$ no real logarithm of u exists. The map $y \mapsto e^y$ has range $(0, \infty)$; solving the transformed linear equation does not remove that range restriction.



Solid: the global IVP solution. Dashed: $\ln(x - 1)$, which is valid only for $x > 1$.

Question 8

On $x > 0$, consider

$$x^2 y' + y = e^{-1/x}.$$

We seek a solution that remains bounded as $x \downarrow 0$.

Tasks

1. Change the independent variable to $t = 1/x$ and define $Y(t) = y(1/t)$. Derive the transformed equation carefully using the chain rule.
2. Solve the transformed equation and recover the full original solution family on $x > 0$.
3. Select the unique solution bounded as $x \downarrow 0$. Find its limit at 0, its limit as $x \rightarrow \infty$, and its value at $x = 1$.
4. Verify it in the original equation and explain how the substitution changes the endpoint at which the boundedness condition is imposed.

Question 8 – Solution

Strategy. A reciprocal change of the independent variable absorbs x^2 , while converting a finite endpoint into an infinite one.

Step 1: Transform the derivative. Since $t = 1/x$, $dt/dx = -1/x^2$. Writing $y(x) = Y(t)$ gives

$$x^2 y' = -Y'(t), \quad -Y' + Y = e^{-t}.$$

Thus $\boxed{Y' - Y = -e^{-t}}$ for $t > 0$. The minus sign comes from the reversal of the independent variable.

Step 2: Solve and return to x . The integrating factor is e^{-t} :

$$(e^{-t}Y)' = -e^{-2t}, \quad Y = \frac{1}{2}e^{-t} + Ce^t.$$

Therefore all solutions on the stated half-line are

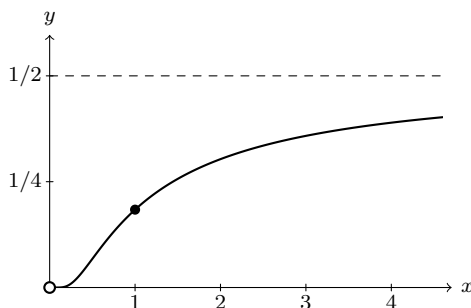
$$\boxed{y(x) = \frac{1}{2}e^{-1/x} + Ce^{1/x}}.$$

Step 3: Apply the endpoint condition. As $x \downarrow 0$, $t \rightarrow +\infty$. The first term tends to zero, whereas any nonzero C makes the second term unbounded. Consequently the unique bounded choice is

$$\boxed{y_b(x) = \frac{1}{2}e^{-1/x}}.$$

Its limits are 0 as $x \downarrow 0$ and $1/2$ as $x \rightarrow \infty$, and $\boxed{y_b(1) = 1/(2e)}$.

Step 4: Verify and interpret the reversal. Differentiation gives $y'_b = e^{-1/x}/(2x^2)$, so $x^2 y'_b + y_b = e^{-1/x}$. The condition near $x = 0$ becomes a condition at $t = \infty$, not at $t = 0$. Conversely, $x \rightarrow \infty$ corresponds to $t \downarrow 0$. The limit at $x = 0$ does not make that point part of the stated domain of the equation.



The bounded solution approaches the excluded point $(0, 0)$ and tends to $1/2$ as $x \rightarrow \infty$.

Question 9

Consider the equation

$$y' = y^2 + (1 - 2x)y + x^2 - x + 1, \quad y(0) = 2.$$

A particular solution is suggested: $y_p(x) = x$.

Tasks

1. Verify the suggested solution and derive the equation for $z = y - y_p$.
2. On a branch with $z \neq 0$, set $u = 1/z$ and derive a linear equation for u .
3. Solve the IVP and find its maximal interval containing 0.
4. Verify the recovered solution through the substitutions, restore the solution excluded by $u = 1/z$, and explain why this reciprocal is applied to the difference from y_p rather than directly to y .

Question 9 – Solution

Strategy. Subtract a known particular solution so that the constant term cancels, then use a reciprocal on the remaining quadratic equation.

Step 1: Verify and subtract the particular solution. For $y_p = x$, the right-hand side is $x^2 + (1 - 2x)x + x^2 - x + 1 = 1 = y'_p$. Now write $y = x + z$. Substitution and cancellation give

$$1 + z' = 1 + z + z^2, \quad \boxed{z' = z + z^2}, \quad z(0) = 2.$$

Step 2: Make the remaining equation linear. For $z \neq 0$, $u = 1/z$ gives

$$u' = -\frac{z'}{z^2} = -u - 1, \quad u(0) = \frac{1}{2}.$$

Solving $u' + u = -1$ yields $u = Ce^{-x} - 1$, and the initial value requires $C = 3/2$.

Step 3: Recover the IVP and its interval. Thus

$$\boxed{y = x + \frac{1}{\frac{3}{2}e^{-x} - 1}, \quad I = (-\infty, \ln(3/2))}.$$

The denominator is positive before its unique zero at $\ln(3/2)$ and gives $y(0) = 2$. At that endpoint $y \rightarrow +\infty$, preventing a finite differentiable extension.

Step 4: Verify and account for the missing solution. Where $u \neq 0$, $z = 1/u$ satisfies $z' = -u'/u^2 = z + z^2$. Thus $y' = 1 + z + z^2$, which is exactly the original right side after $y = x + z$. The excluded case $z \equiv 0$ restores $\boxed{y = x}$, a global solution with a different initial value.

The original equation contains a nonzero term independent of y . A direct reciprocal $w = 1/y$ would retain a quadratic term in w and would not be linear. Subtracting the known solution first cancels that obstruction. This two-step substitution is the useful structure, not the reciprocal alone.

Question 10

For a homogeneous equation $y' = F(y/x)$ on $x > 0$, introduce the logarithmic independent variable $t = \ln x$ and the ratio $v(t) = y(e^t)/e^t$.

We want F to be a polynomial of degree at most two such that, in the transformed equation, $v = 1$ and $v = 3$ are equilibrium values and $dv/dt = -1$ when $v = 2$.

Tasks

1. Derive the equation for dv/dt in terms of F and v .
2. Determine the unique polynomial F satisfying the design requirements.
3. Solve the resulting original IVP with $y(1) = 2$, showing the transformed integration and recovery of y .
4. Find the limits of y/x as $x \downarrow 0$ and as $x \rightarrow \infty$. Identify the original solutions represented by the equilibrium ratios and explain why they are not constant functions y . Decide whether the selected original $y(x)$ increases or decreases.

Question 10 – Solution

Strategy. The logarithmic independent variable turns multiplication of x into translation of t and removes the factor x from the ratio equation.

Step 1: Derive and design the transformed equation. Writing $y = xv(\ln x)$ gives $y' = v + dv/dt$, so

$$\frac{dv}{dt} = F(v) - v.$$

This polynomial has roots 1, 3 and degree at most two, hence equals $k(v-1)(v-3)$. Its value -1 at $v = 2$ forces $k = 1$. Therefore

$$\boxed{F(v) = v^2 - 3v + 3}, \quad v' = (v-1)(v-3).$$

Step 2: Solve the selected trajectory. The data give $v(0) = 2$, between the equilibria. Separation yields

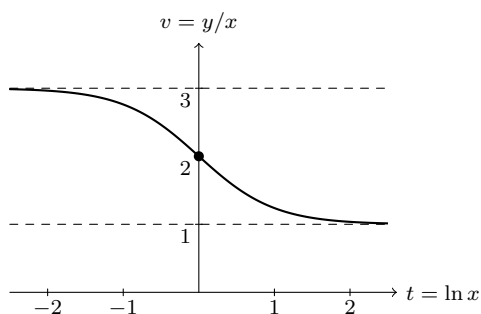
$$\frac{1}{2} \ln \left| \frac{v-3}{v-1} \right| = t + C, \quad C = 0.$$

The ratio is negative on the selected branch, so $(v-3)/(v-1) = -e^{2t}$. Thus

$$v = 1 + \frac{2}{1 + e^{2t}}, \quad \boxed{y = x + \frac{2x}{1 + x^2}, \quad x > 0}.$$

Directly $dv/dt = -4e^{2t}/(1 + e^{2t})^2 = (v-1)(v-3)$, so the chain rule verifies the original equation and $y(1) = 2$.

Step 3: Interpret the limits and equilibria. As $x \downarrow 0$, $t \rightarrow -\infty$ and $y/x \rightarrow 3$; as $x \rightarrow \infty$, $t \rightarrow \infty$ and $y/x \rightarrow 1$. The constant ratios give $\boxed{y = x, y = 3x}$, both valid on $x > 0$. They are straight lines, not constant original functions: constancy applies to the ratio in the new coordinates. Moreover, $y' = F(v) = (v-3/2)^2 + 3/4 > 0$, so the original y increases even though its ratio v decreases.



The transformed ratio decreases from 3 to 1; the dashed levels represent $y = 3x$ and $y = x$.

Question 1

Consider the initial-value problem

$$y' = y^2, \quad y(0) = 2.$$

A student argues: “The right-hand side and its derivative with respect to y are continuous everywhere, so the solution must exist for every real x .”

Tasks

1. State what local existence and uniqueness actually guarantee for this IVP, and identify the gap in the student’s argument.
2. Solve the IVP and verify the initial value and differential equation.
3. Find its maximal open interval of validity containing 0, and prove that a classical solution cannot extend it past its finite endpoint.
4. The same rational expression is defined on another open interval. Explain why that branch is not a continuation of this IVP solution.

Here a classical solution is a real, continuously differentiable function satisfying the equation at every point of its interval. A maximal interval cannot be enlarged while extending the same solution of the given equation.

Question 1 – Solution

Strategy. Smoothness of the differential equation gives a local theorem; the explicit solution reveals whether its values remain finite.

Step 1: Separate local and global conclusions. Here $f(x, y) = y^2$ and $f_y(x, y) = 2y$ are continuous on the whole plane. The local theorem guarantees a solution on some open interval around 0 and uniqueness of solutions through $(0, 2)$ on their common interval. It does not guarantee a solution on the entire x -axis: growth in y can produce a finite-time blow-up.

Step 2: Solve and verify. Near the initial point $y \neq 0$, so separation gives

$$-\frac{1}{y} = x + C, \quad C = -\frac{1}{2}, \quad \boxed{y(x) = \frac{2}{1 - 2x}}.$$

Directly, $y(0) = 2$ and $y' = 4/(1 - 2x)^2 = y^2$. The expression therefore defines the selected solution at every $x < 1/2$.

Step 3: Establish maximality. The component of the formula's domain containing 0 is

$$\boxed{I = (-\infty, 1/2)}.$$

As $x \uparrow 1/2$, $y(x) \rightarrow +\infty$. Any extension to a larger interval would include $1/2$ as an interior point and would have a finite, continuous value there. That contradicts this limit. There is no finite obstruction to the left.

Step 4: Distinguish the disconnected branch. The expression also solves the differential equation on $(1/2, \infty)$, where it is negative. That interval does not contain the initial point. The union of the two components is not an interval, and no classical solution can connect them through the pole. An algebraic formula on both sides of a singularity is not a continuation across it.

Question 2

For a real parameter a , consider the equation in the form written:

$$y' + \frac{2x}{x^2 - 9}y = \frac{1}{x^2 - 9}, \quad y(0) = a.$$

The original equation is defined only for $x \neq -3, 3$.

Tasks

1. Use the linear existence theorem to identify an interval on which a unique solution is guaranteed for every a .
2. Find and verify the solution formula in terms of a .
3. Determine which values of a make a singularity of that formula removable, and compute the corresponding finite limit.
4. Determine the maximal interval for the original IVP for every a . Explain whether multiplying the equation by $x^2 - 9$ changes what it means to extend a solution through $x = \pm 3$.

Question 2 – Solution

Strategy. Keep the domain of the given equation separate from the domain of a simplified formula or a multiplied equation.

Step 1: Locate the coefficient interval. Both coefficients are continuous on $(-3, 3)$, the largest coefficient interval containing 0. The linear theorem guarantees a unique solution throughout that interval for every real a .

Step 2: Integrate and verify. On this interval, multiplication and the product rule yield

$$((x^2 - 9)y)' = 1, \quad (x^2 - 9)y = x - 9a.$$

Thus

$$y(x) = \frac{x - 9a}{x^2 - 9}.$$

The value at 0 is a , and differentiating $(x^2 - 9)y = x - 9a$ verifies the differential equation whenever $x \neq \pm 3$.

Step 3: Examine cancellation. At $x = 3$, the numerator vanishes exactly when $a = 1/3$. Then

$$y = \frac{1}{x + 3} \quad (x \neq \pm 3), \quad \lim_{x \rightarrow 3} y = \frac{1}{6}.$$

At $x = -3$, cancellation occurs exactly when $a = -1/3$. Then

$$y = \frac{1}{x - 3} \quad (x \neq \pm 3), \quad \lim_{x \rightarrow -3} y = -\frac{1}{6}.$$

No value of a cancels both factors. For other parameter values the corresponding endpoint is a pole.

Step 4: Use the original domain. For every a , the answer remains $I = (-3, 3)$. Even a finite removable limit does not define the original coefficients at an excluded point.

The multiplied equation $(x^2 - 9)y' + 2xy = 1$ is defined at $x = \pm 3$ and imposes algebraic conditions there. It is equivalent to the original equation only away from those points. A canceled formula can extend through one endpoint as a solution of the multiplied equation, but that is an extension for a different domain of definition.

Question 3

Consider

$$y' = \frac{e^{-y}}{x-1}, \quad y(2) = \ln 2,$$

with $x \neq 1$ and real y .

Tasks

1. Use $u = e^y$ to solve the IVP, explicitly enforcing the range of this substitution.
2. Determine the maximal interval containing 2; compare its left endpoint with the singular point of the coefficient $1/(x-1)$.
3. Verify the solution and calculate its limiting behavior at both ends of its maximal interval.
4. Explain why neither continuing the linear transformed solution through a zero nor replacing $\ln u$ by $\ln |u|$ produces a continuation of the original solution.

Question 3 – Solution

Strategy. The exponential substitution is invertible only for $u > 0$. Its range can restrict the interval before a coefficient singularity is reached.

Step 1: Transform and invert. On the component $x > 1$ containing 2,

$$u' = e^y y' = \frac{1}{x-1}, \quad u = \ln(x-1) + C.$$

Since $u(2) = 2$, $C = 2$. Therefore

$$y = \ln(2 + \ln(x-1))$$

is a real solution only where $2 + \ln(x-1) > 0$.

Step 2: Find the interval. The positivity condition is $x > 1 + e^{-2}$. Let $\alpha = 1 + e^{-2}$; then

$$I = (\alpha, \infty).$$

In particular, $1 < \alpha < 2$. The solution fails before reaching $x = 1$, even though the original right-hand side is smooth at every finite point (α, y) .

Step 3: Verify and analyze the ends. Writing $u = 2 + \ln(x-1) > 0$ gives

$$y' = \frac{1}{(x-1)u} = \frac{e^{-y}}{x-1}, \quad y(2) = \ln 2.$$

As $x \downarrow \alpha$, $u \downarrow 0$ and $y \rightarrow -\infty$, preventing a finite continuous extension. As $x \rightarrow \infty$, $u \rightarrow \infty$ and $y \rightarrow \infty$, but this occurs only at an infinite endpoint and does not shorten the interval.

Step 4: Reject an invalid inverse. The transformed function continues on $(1, \infty)$ but becomes negative on $(1, \alpha)$. No real y can have $e^y = u \leq 0$.

On that negative portion, $z = \ln |u|$ would satisfy $z' = 1/((x-1)u) < 0$, whereas $e^{-z}/(x-1) = 1/((x-1)|u|) > 0$. Thus the absolute-value replacement fails the original equation and cannot repair the lost exponential range.

Question 4

Consider the real IVP

$$y' = -\frac{x}{y}, \quad y(0) = 2,$$

whose equation is defined only when $y \neq 0$.

Tasks

1. Obtain an implicit relation and select the explicit branch determined by the initial condition.
2. Find the maximal interval and the limits of both y and y' at its finite endpoints.
3. Decide whether either endpoint can be included, or whether switching to the other square-root branch gives a continuation as a classical solution $y(x)$.
4. Sketch the selected solution and mark its excluded endpoints. Explain why bounded values of y do not guarantee continuation in this example.

Question 4 – Solution

Strategy. Track the forbidden value $y = 0$ and the slope, rather than checking only whether the solution stays bounded.

Step 1: Select the branch. Multiplying by y and integrating gives

$$yy' = -x, \quad x^2 + y^2 = 4.$$

Continuity and $y(0) = 2$ select

$$y = \sqrt{4 - x^2}.$$

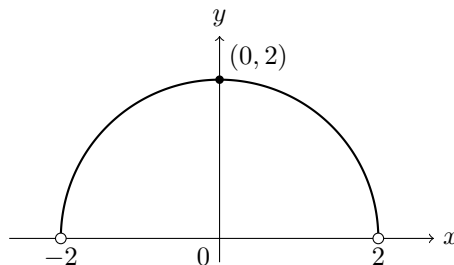
Its positive branch has $y' = -x/\sqrt{4 - x^2}$ and directly satisfies the IVP for $-2 < x < 2$.

Step 2: Test both endpoints. Thus $I = (-2, 2)$, and

	$x \downarrow -2$	$x \uparrow 2$
$y(x)$	0^+	0^+
$y'(x)$	$+\infty$	$-\infty$

At either endpoint the original right-hand side is undefined, and a continuously differentiable extension would also require a finite derivative.

Step 3: Exclude branch switching. The negative square root fails the initial condition. Joining the two semicircles at an endpoint still passes through $y = 0$, and the circle has no real points with $|x| > 2$. Following the circle as a parametrized curve is not an extension as a differentiable graph $y(x)$ on a larger interval.



Step 4: Identify the obstruction. Although $0 < y \leq 2$, the graph approaches the boundary of the equation's domain. Boundedness alone does not keep it inside a region where the differential equation is defined and regular. The open circles indicate excluded endpoints, not additional solution values.

Question 5

Consider

$$y' = \frac{1}{3y^2 - 1}, \quad y(0) = 0,$$

with $y \neq \pm 1/\sqrt{3}$.

Tasks

1. Derive an implicit equation and identify the portion that defines the IVP's branch as a function of x .
2. Determine the exact maximal interval containing 0, using monotonicity rather than a cubic-root formula.
3. Find the one-sided limits of y and y' at both endpoints and justify maximality.
4. Sketch the selected branch, and explain why other roots of the same cubic cannot be spliced onto it to extend this classical solution.

Question 5 – Solution

Strategy. Regard x as a function of y . Its turning points identify where the selected inverse branch stops being a regular graph.

Step 1: Integrate without solving the cubic. The chain rule gives

$$\frac{d}{dx}(y^3 - y) = (3y^2 - 1)y' = 1, \quad \boxed{x = y^3 - y}.$$

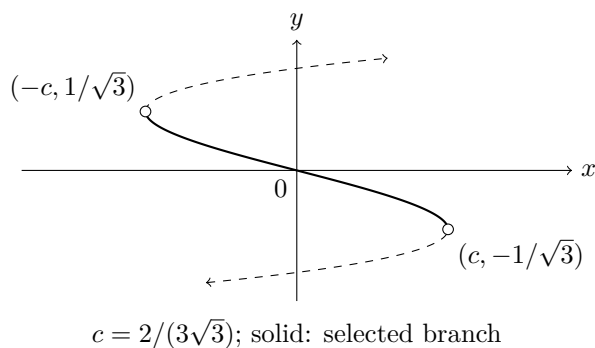
The initial value sets the integration constant to zero. Put $h(y) = y^3 - y$. On $-1/\sqrt{3} < y < 1/\sqrt{3}$, $h'(y) < 0$, so h is strictly decreasing and has a differentiable inverse there.

Step 2: Map the branch endpoints. Let $c = 2/(3\sqrt{3})$. Then $h(1/\sqrt{3}) = -c$ and $h(-1/\sqrt{3}) = c$. Hence the inverse through $(0, 0)$ exists exactly on

$$\boxed{I = (-c, c) = \left(-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right)}.$$

Differentiating its defining relation recovers $y' = 1/(3y^2 - 1)$ and verifies the original equation on this branch.

Step 3: Analyze the folds. As $x \downarrow -c$, $y \rightarrow 1/\sqrt{3}$; as $x \uparrow c$, $y \rightarrow -1/\sqrt{3}$. In both cases $3y^2 - 1 \rightarrow 0^-$, so $y' \rightarrow -\infty$. Both finite limiting values are excluded by the original equation.



Step 4: Reject a change of root. The dashed pieces show other parts of the algebraic curve. A continuous switch at a fold must include an excluded value of y with unbounded slope; a switch to a different root at that x would be discontinuous. Neither gives a classical extension. The cubic relation alone does not specify the IVP's interval or branch.

Question 6

Consider the IVP

$$y' = 3|y|^{2/3}, \quad y(0) = 0.$$

For finite real numbers $a \leq 0 \leq b$, define

$$Y_{a,b}(x) = \begin{cases} (x-a)^3, & x < a, \\ 0, & a \leq x \leq b, \\ (x-b)^3, & x > b. \end{cases}$$

Tasks

1. Verify the equation and continuous differentiability of $Y_{a,b}$, including both joining points.
2. Give two distinct global solutions satisfying the initial condition and determine their maximal intervals.
3. Explain why a restriction of $Y_{-1,1}$ to $(-1/2, 1/2)$ is not maximal, even though it solves the IVP there.
4. Sketch $Y_{-1,1}$ and identify precisely which local uniqueness hypothesis fails. Does a maximal interval imply a unique solution?

Question 6 – Solution

Strategy. Check continuation at the joins directly. The length of a solution's interval and uniqueness of the solution are different questions.

Step 1: Verify all three pieces and the joins. For $x < a$,

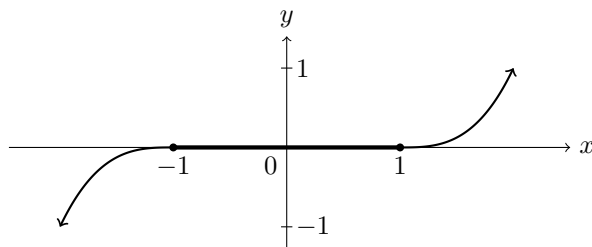
$$Y'_{a,b} = 3(x-a)^2 = 3|(x-a)^3|^{2/3},$$

and the same computation holds with b for $x > b$. On the middle piece both sides vanish.

At $x = a$, the values from both sides are 0, and the one-sided derivatives tend to 0. The derivative at the join itself is 0 by the difference quotient. The same holds at b , including the case $a = b = 0$. Thus $Y_{a,b}$ is continuously differentiable on $\mathbb{R}$, satisfies the equation everywhere, and has $Y_{a,b}(0) = 0$.

Step 2: Exhibit distinct maximal solutions. Two examples are $y(x) = 0$ for all x and $y(x) = x^3 = Y_{0,0}(x)$. Their maximal intervals are both $\boxed{\mathbb{R}}$: no larger real interval exists. The function $Y_{-1,1}$ is a third global solution.

Step 3: Separate a restriction from a maximal solution. On $(-1/2, 1/2)$, $Y_{-1,1} = 0$, but it extends as $Y_{-1,1}$ to the whole real line. Its restriction is therefore not maximal. That restriction can also be extended as the identically zero solution; an extension need not be unique here.



Solid graph: $Y_{-1,1}$; thick segment: $y = 0$

Step 4: Locate the failed uniqueness condition. The right-hand side is continuous, but it is not locally Lipschitz in y at 0:

$$\frac{|3|y|^{2/3} - 0|}{|y - 0|} = 3|y|^{-1/3} \rightarrow \infty.$$

Consequently the usual local uniqueness hypothesis is unavailable at the initial point. The explicit examples prove nonuniqueness, even though every exhibited global solution has the same maximal interval $\mathbb{R}$.

Question 7

Let p be any real number. Consider the family of IVPs

$$y' = y(1 - y), \quad y(0) = p.$$

Focus on intervals of validity; a full equilibrium or phase-line analysis is not required.

Tasks

1. Derive a solution formula in terms of p , preserving the cases lost by division in separation.
2. Classify the maximal interval containing 0 for every real p . Give any finite endpoint explicitly.
3. Determine the direction and sign of blow-up whenever such an endpoint exists.
4. Verify the formula and explain why the interval depends on p even though the differential equation is smooth on the whole (x, y) -plane.

Question 7 – Solution

Strategy. A denominator zero is relevant only in the connected component containing the initial point. Preserve constant solutions before separating.

Step 1: Obtain a parameter formula. The constant solutions are $y = 0, 1$. Otherwise separation gives $\ln |y/(1-y)| = x + C$. Applying the initial value yields a formula that also includes both constants:

$$y(x) = \frac{pe^x}{1-p+pe^x}.$$

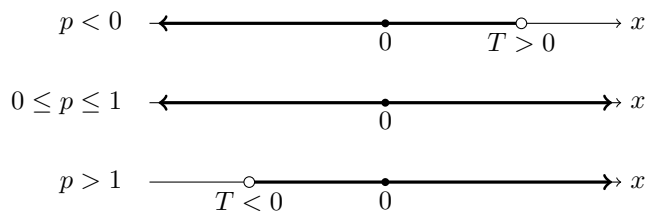
Its denominator $D(x)$ satisfies $D(0) = 1$. A zero requires $e^x = (p-1)/p > 0$, which occurs exactly when $p < 0$ or $p > 1$.

Step 2: Select the component containing zero. When it exists, set $T = \ln((p-1)/p)$. The complete classification is

initial value	endpoint location	maximal interval
$p < 0$	$T > 0$	$(-\infty, T)$
$0 \leq p \leq 1$	no finite endpoint	$\mathbb{R}$
$p > 1$	$T < 0$	(T, ∞)

For $0 < p < 1$, both terms of $D = 1 - p + pe^x$ are positive. The cases $p = 0, 1$ are global constants.

Step 3: Determine the endpoint behavior. If $p < 0$, then $D \downarrow 0^+$ as $x \uparrow T$, while $pe^x \rightarrow p-1 < 0$, so $y \rightarrow -\infty$. If $p > 1$, then $D \downarrow 0^+$ as $x \downarrow T$, while $pe^x \rightarrow p-1 > 0$, so $y \rightarrow +\infty$. Neither pole is removable.



Interval diagrams; endpoint positions schematic

Step 4: Verify and interpret. With $N = pe^x$, $N' = D' = N$, and

$$y' = \frac{N(D-N)}{D^2} = y(1-y), \quad y(0) = p.$$

Smoothness gives local existence and uniqueness. It does not prevent the finite-time blow-up found for $p \notin [0, 1]$; the initial value determines whether the selected solution encounters a pole.

Question 8

Consider the equation in its original, undivided form:

$$xy' = 2y, \quad y(1) = 1.$$

It is defined for all real x , including $x = 0$. A classical solution must be continuously differentiable and satisfy this equation at every point of its open interval.

Tasks

1. Solve on $x > 0$ and explain why dividing by x cannot by itself settle continuation through 0 for the original equation.
2. Find every global classical solution of the original IVP.
3. Check both the differential equation and continuous differentiability at 0 for your full family.
4. Determine the maximal interval of each such solution. Explain how uniqueness on $x > 0$ can coexist with nonunique global continuation, and why this differs from an equation explicitly defined only for $x \neq 0$.

Question 8 – Solution

Strategy. The original equation remains meaningful at the vanishing leading coefficient. Solve on the two half-lines and test the join in that original equation.

Step 1: Solve where division is allowed. For $x \neq 0$, the equation gives $(y/x^2)' = 0$. The initial condition therefore forces $y = x^2$ throughout $x > 0$. On $x < 0$, the general solution is $y = Cx^2$, with an independent real constant C .

The divided equation $y' = 2y/x$ has no meaning at 0, but the original equation does: it requires $y(0) = 0$. A singularity introduced by division is not automatically an obstruction for the original problem.

Step 2: List all global solutions. Every global candidate must have the form

$$y_C(x) = \begin{cases} Cx^2, & x < 0, \\ 0, & x = 0, \\ x^2, & x > 0, \end{cases} \quad C \in \mathbb{R}.$$

The half-line solutions and the equation at 0 leave no other possible global classical solutions.

Step 3: Verify the join. For every finite C , y_C is continuous at 0. The difference quotient there is Ch for $h < 0$ and h for $h > 0$, so $y'_C(0) = 0$. Moreover,

$$y'_C(x) = 2Cx \quad (x < 0), \quad y'_C(x) = 2x \quad (x > 0)$$

tends to 0 from both sides. Thus y_C is continuously differentiable. The identity $xy'_C = 2y_C$ holds on each half-line and reduces to $0 = 0$ at 0. Also $y_C(1) = 1$.

Step 4: Resolve maximality and uniqueness. Each y_C has maximal interval $\boxed{\mathbb{R}}$. The unique positive-half-line solution extends through 0 in infinitely many ways because the original equation loses its derivative term there. The usual normal-form uniqueness theorem cannot cross that degeneracy.

If the problem had instead specified $y' = 2y/x$ only for $x \neq 0$, its IVP interval containing 1 would be $(0, \infty)$, even for the polynomial formula. Which equation and domain were actually given matters.

Question 9

Define a real function on the entire real line by

$$f(x) = \begin{cases} -1, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Consider $y' = f(x)$ with $y(-1) = 0$. Require a classical solution: y is continuously differentiable on an open interval and the equation holds at every point.

Tasks

1. Find the IVP solution on $x < 0$ and its maximal interval containing -1 .
2. Compute the finite left-hand limits of y and y' at 0.
3. Find the continuous function on $\mathbb{R}$ that agrees with this solution on $x < 0$ and solves the differential equation on $x > 0$. Decide whether it is a classical extension through 0.
4. Explain why this example does not contradict a continuation theorem requiring a continuous right-hand side. Would changing only $f(0)$ repair the obstruction?

Question 9 – Solution

Strategy. A formula obtained by integrating on either side of a jump need not be differentiable at the joining point.

Step 1: Solve on the initial component. For $x < 0$, $y' = -1$, so the initial condition gives

$$\boxed{y = -x - 1}.$$

It solves the equation at every negative x . Its maximal classical interval containing -1 is $\boxed{(-\infty, 0)}$, as the join test below shows.

Step 2: Compute the finite endpoint limits. As $x \uparrow 0$, $y(x) \rightarrow -1$ and $y'(x) \rightarrow -1$. Neither the solution nor its derivative blows up. The obstruction will be the change of slope imposed immediately to the right.

Step 3: Test the unique continuous gluing. On $x > 0$, any solution is $x + C$. Continuity with the left branch forces $C = -1$ and the value $y(0) = -1$. The only continuous gluing is therefore

$$Y(x) = |x| - 1.$$

Its left difference quotient at 0 is -1 and its right difference quotient is 1 . Consequently $Y'(0)$ does not exist. It is not a classical solution of the equation on any interval containing 0.

Any larger interval extending the negative-half-line solution would include points on both sides of 0 and would require this same gluing. Thus no classical extension is possible, proving the claimed maximality despite the finite limits.

Step 4: Identify the missing hypothesis. Here f is defined everywhere but is discontinuous at 0. A continuation result that requires continuity of the right-hand side near the endpoint cannot be applied. Domain membership and bounded values alone are insufficient when that hypothesis fails.

Changing $f(0)$ to any single real value cannot reconcile the unequal one-sided derivatives forced by $f = -1$ to the left and $f = 1$ to the right. The same corner remains. The function Y solves the equation away from the join, but the question requires the equation at every point of a classical interval.

Question 10

Design an initial-value problem of the form

$$y' = (Ax + B)y^2, \quad y(0) = 1,$$

where A, B are real constants, so that its solution has exactly the maximal interval $(-2, 3)$ and tends to $+\infty$ at both finite endpoints.

Tasks

1. Use the reciprocal variable $u = 1/y$ to determine A and B from the prescribed endpoints.
2. Recover the solution and prove that it is positive and has exactly the requested maximal interval.
3. Verify the IVP and prove that your coefficient pair is the only affine choice with these properties.
4. Find the solution's minimum on its interval and sketch the curve with its endpoint asymptotes. Explain why smooth coefficients do not force a global solution here.

Question 10 – Solution

Strategy. For a quadratic-growth equation, the reciprocal solution is a quadratic polynomial. Put its zeros at the desired endpoints.

Step 1: Determine the reciprocal polynomial. While $y \neq 0$,

$$u' = -(Ax + B), \quad u(0) = 1, \quad u = 1 - Bx - \frac{A}{2}x^2.$$

Blow-up at both prescribed endpoints requires $u(-2) = u(3) = 0$. A quadratic with these roots and value 1 at 0 must be

$$u = \frac{(x+2)(3-x)}{6} = 1 + \frac{x}{6} - \frac{x^2}{6}.$$

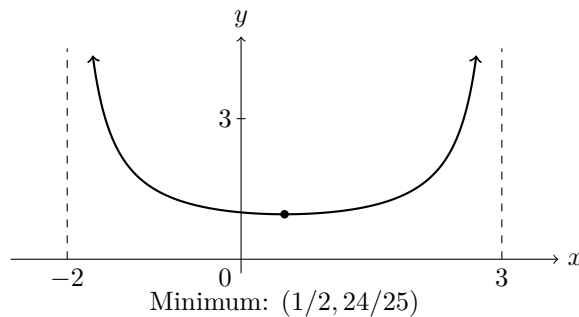
Thus $A = 1/3$, $B = -1/6$ and

$$y = \frac{6}{(x+2)(3-x)}, \quad I = (-2, 3).$$

Step 2: Check the interval and uniqueness of the design. Both denominator factors are positive on I . At either endpoint the denominator tends to 0^+ , so $y \rightarrow +\infty$ and no finite continuous extension exists.

Also $y' = -u'/u^2 = ((2x-1)/6)y^2$ and $y(0) = 1$. For any proposed affine pair, a solution through 1 cannot cross the zero solution, by local uniqueness for the smooth right-hand side. Hence its reciprocal on the proposed interval has exactly the polynomial form above. The two endpoint zeros and $u(0) = 1$ determine it uniquely, proving uniqueness of the coefficient pair.

Step 3: Locate the minimum. The denominator is $25/4 - (x - 1/2)^2$, so it is largest at $x = 1/2$. Therefore $y_{\min} = 24/25$ there. Equivalently, y' changes from negative to positive at $1/2$.



Step 4: Interpret the construction. The coefficients and the right-hand side are smooth everywhere, but the solution grows without bound in both finite endpoint directions. Smoothness ensures local solvability and uniqueness, not global existence for this nonlinear equation.

Question 1

A tank initially holds 100 L of well-mixed solution containing 10 kg of salt. Solution containing 0.20 kg/L enters at 3 L/min, and the mixture leaves at 1 L/min. The tank has capacity 200 L. Assume instantaneous mixing, additive liquid volumes, and no salt precipitation or other losses. Let t be minutes after the flows begin.

Tasks

1. Derive initial-value models for the liquid volume $V(t)$ and salt mass $S(t)$, including units for all rate terms and the operating time before overflow.
2. Derive and solve an equation for the concentration $c(t) = S(t)/V(t)$. Explain why c' is not simply S'/V .
3. Find the first time the concentration reaches 0.15 kg/L and decide whether this happens before the tank reaches capacity.
4. Find the salt mass when the tank first becomes full. Sketch the concentration over the modeled operation and explain why continuing the same volume formula beyond that time changes the physical problem.

Question 1 – Solution

Strategy. Write separate liquid and salt balances, then use the quotient rule to model concentration in a changing volume.

Step 1: Balance volume and salt. The volume rate is $V' = 3 - 1 = 2$ L/min, so $V = 100 + 2t$ L. Capacity is reached at $t = 50$ min. Up to that time,

$$S' = 0.60 - \frac{S}{100 + 2t}, \quad S(0) = 10.$$

Both terms have units kg/min: salt enters at $(3)(0.20)$ and leaves at $(1)(S/V)$. The pre-overflow model applies for $0 \leq t < 50$, with a continuous value at 50.

Step 2: Derive the concentration equation. Since $S = cV$, $S' = Vc' + cV'$. Hence

$$(100 + 2t)c' = 0.60 - 3c, \quad c(0) = 0.10.$$

Using the integrating factor $(100 + 2t)^{3/2}$ and $c(0) = 0.10$ gives

$$c(t) = 0.20 - 0.10(100/(100 + 2t))^{3/2}.$$

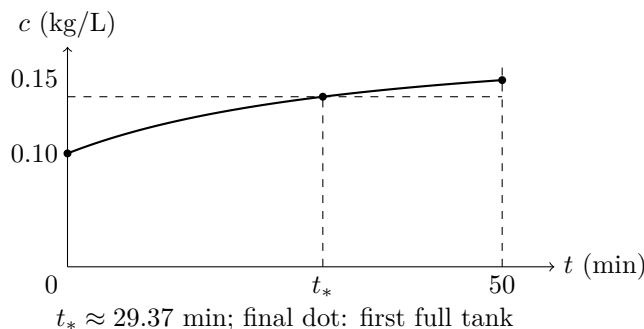
Indeed $c' = 3(0.20 - c)/V$, verifying the concentration balance. Omitting cV' would ignore the dilution caused by increasing volume.

Step 3: Locate the target and capacity values. The concentration increases strictly. Setting $c = 0.15$ gives

$$t_* = 50(2^{2/3} - 1) \approx 29.37 \text{ min} < 50 \text{ min}.$$

At capacity, $c(50) = 0.20 - 0.10/2^{3/2}$, so

$$S(50) = 200c(50) = 40 - 5\sqrt{2} \approx 32.93 \text{ kg}.$$



Step 4: Respect the operating boundary. The final point is the instant the tank becomes full. Beyond it, continued inflow would require overflow or a changed flow rate. Either changes the volume and salt balances, so the formula is not a model of the same operation after 50 min.

Question 2

An object is placed in a controlled chamber. Its temperature is initially 80°C , while the chamber temperature is $T_a(t) = 20 + 2t$ degrees Celsius, with t in minutes. Assume the object has a uniform temperature and obeys Newton's law of cooling with constant coefficient $k = 0.10 \text{ min}^{-1}$ throughout the experiment. The chamber warming is prescribed and unaffected by the object.

Tasks

1. Formulate the temperature IVP, including the sign of the heat-transfer term.
2. Solve it and verify the initial temperature and differential equation.
3. Find when the object stops cooling and determine its minimum temperature. Explain how the object can subsequently warm under the same law.
4. Sketch the object and chamber temperatures. Find their limiting difference and explain why their temperatures do not become equal permanently.

Question 2 – Solution

Strategy. Model the difference between the object and its time-dependent surroundings; the ambient temperature is not a constant equilibrium value.

Step 1: Form and solve the IVP. Newton's law gives

$$T' = -0.10(T - 20 - 2t), \quad T(0) = 80.$$

The right side is in degrees Celsius per minute. The object cools when it is hotter than the chamber and warms when it is colder. An integrating factor $e^{t/10}$ yields

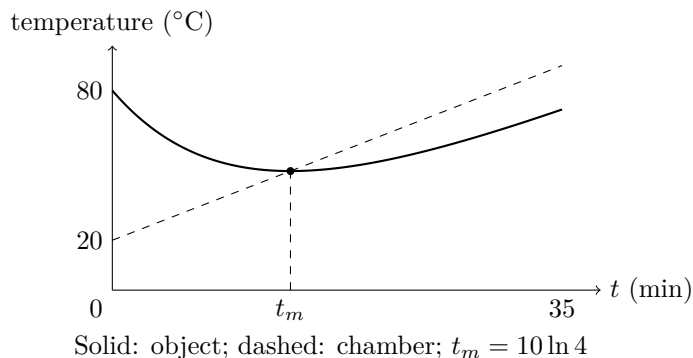
$$T(t) = 2t + 80e^{-t/10}, \quad t \geq 0.$$

Its derivative is $2 - 8e^{-t/10}$, which equals $-0.10(T - T_a)$, and its initial value is 80.

Step 2: Identify the turning time. The derivative vanishes when $e^{-t/10} = 1/4$, giving

$$t_m = 10 \ln 4 \approx 13.86 \text{ min}, \quad T_{\min} = 20 + 20 \ln 4 \approx 47.73^\circ\text{C}.$$

Since $T'' = 0.8e^{-t/10} > 0$, the derivative changes from negative to positive exactly once. At this time $T = T_a$; afterward the rising chamber is warmer than the object, so the heat-transfer term becomes positive.



Step 3: Interpret the long-time lag. The signed temperature difference is

$$T - T_a = 80e^{-t/10} - 20 \longrightarrow -20^\circ\text{C}.$$

The object eventually warms at almost $2^\circ\text{C}/\text{min}$. A temperature deficit of 20°C supplies that rate because $-0.10(-20) = 2$. Permanent equality would instead give zero heat-transfer rate, inconsistent with following an ambient temperature that keeps rising. The long-time conclusion is conditional on the stated linear ambient schedule and constant cooling coefficient remaining valid.

Question 3

A specimen contains a stable material of unknown mass B and an active material of mass $R(t)$ that disappears from the measured specimen at a rate proportional to R . Assume the stable material is unaffected and all disappearance products leave the measured mass. Thus the measured mass is $M(t) = B + R(t)$. Measurements, treated as exact, are

$$M(0) = 100 \text{ g}, \quad M(1) = 70 \text{ g}, \quad M(2) = 55 \text{ g},$$

where t is in days. Assume $B \geq 0$, $R(0) > 0$, and a positive constant disappearance coefficient.

Tasks

1. Formulate the active-mass IVP and the resulting measurement model.
2. Determine uniquely the stable mass, initial active mass, and disappearance coefficient from the data.
3. Find the active material's half-life and the time at which only 5 g of active material remain.
4. A technician fits exponential decay directly to the total mass using the first two measurements. Test that model against the third measurement and explain the modeling error.

Question 3 – Solution

Strategy. Separate the unchanging baseline from the decaying component. Differences of equally spaced measurements remove the baseline.

Step 1: Write the component and observation models. The active mass satisfies $R' = -kR$, where $k > 0$ has units day^{-1} . If $R(0) = R_0$, then

$$R = R_0 e^{-kt}, \quad M = B + R_0 e^{-kt}.$$

The stable material contributes to the measurement but not to the loss rate.

Step 2: Identify the parameters. Let $r = e^{-k}$, where one day is the time step, so $0 < r < 1$. The measured successive differences satisfy

$$-30 = R_0(r - 1), \quad -15 = R_0 r(r - 1).$$

Their ratio gives $r = 1/2$. Therefore

$$\boxed{R_0 = 60 \text{ g}, \quad B = 40 \text{ g}, \quad k = \ln 2 \text{ day}^{-1}}, \quad \boxed{M(t) = 40 + 60 2^{-t}}.$$

These values reproduce 100, 70, 55 g. They are unique under the assumptions because the nonzero first difference fixes r , then R_0 , then B . Also $R' = -kR$ verifies the rate law.

Step 3: Compute active-mass times. The active half-life is $\ln 2/k = \boxed{1 \text{ day}}$. The condition $R = 5$ gives $60 2^{-t} = 5$, hence

$$\boxed{t = \frac{\ln 12}{\ln 2} \approx 3.585 \text{ days}}.$$

The active mass is below 5 g after that time. The total mass then is 45 g, not 5 g.

Step 4: Test the incorrect observation model. A single exponential through the first two total-mass measurements is $\widetilde{M}(t) = 100(0.7)^t$. It predicts $\widetilde{M}(2) = 49$ g, contradicting the measured 55 g. It wrongly assigns the stable component the same disappearance law as the active material and predicts a zero limiting total mass instead of 40 g.

The model is intended for $t \geq 0$. Its inference depends on exact data, constant k , and a truly constant baseline; noisy data would require an estimation method rather than exact identification from three values.

Question 4

A 2 kg test package is released from rest. Take downward as positive and use $g = 10 \text{ m/s}^2$. Model drag as a force $-bv$ opposing its downward velocity v . Before a parachute opens at $t = 4 \text{ s}$, $b = 2 \text{ kg/s}$; afterward, $b = 10 \text{ kg/s}$. Assume the opening changes the drag coefficient instantly but delivers no impulse, and the package remains airborne through $t = 6 \text{ s}$.

Tasks

1. Derive the velocity equation in each phase from Newton's second law. State the initial and matching conditions.
2. Solve for $v(t)$ through 6 s and calculate the velocity immediately before and after opening.
3. Find the total downward distance traveled during the first 6 s.
4. Find the one-sided accelerations at opening and sketch the velocity. Explain why velocity is continuous although acceleration jumps, and state where a classical phase equation applies.

Question 4 – Solution

Strategy. Balance forces in each phase and carry velocity continuously across a coefficient change; integrate velocity to obtain displacement.

Step 1: Model and match the phases. Newton's law gives $2v' = 20 - bv$, with forces in newtons. Thus $v' = 10 - v$ before opening and $v' = 10 - 5v$ afterward. The initial value is $v(0) = 0$, and no impulse means $v(4^-) = v(4^+) = v_4$.

Step 2: Solve both initial-value problems. The solutions are

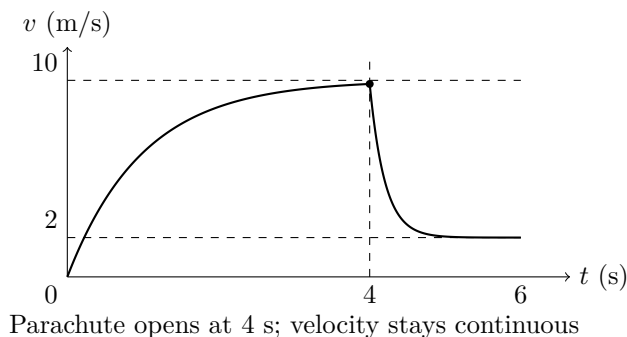
$$v(t) = \begin{cases} 10(1 - e^{-t}), & 0 \leq t \leq 4, \\ 2 + (v_4 - 2)e^{-5(t-4)}, & 4 \leq t \leq 6, \end{cases} \quad v_4 = 10(1 - e^{-4}) \approx 9.817 \text{ m/s}.$$

Each expression satisfies its phase equation, and both give v_4 at the switch. The phase terminal speeds are 10 and 2 m/s, respectively.

Step 3: Integrate the distance. Velocity stays positive, so downward displacement equals distance traveled:

$$d = 10(3 + e^{-4}) + 4 + \frac{8 - 10e^{-4}}{5}(1 - e^{-10}) \approx 35.75 \text{ m}.$$

The first term integrates the pre-opening velocity over $[0, 4]$; the remaining terms integrate the post-opening velocity over $[4, 6]$.



Step 4: Examine the acceleration jump. The one-sided accelerations are

$$v'(4^-) = 10e^{-4} \approx 0.183 \text{ m/s}^2, \quad v'(4^+) = -40 + 50e^{-4} \approx -39.08 \text{ m/s}^2.$$

Changing a finite force changes acceleration without an instantaneous velocity jump. The velocity is continuous and piecewise differentiable, but is not differentiable at the idealized switch. Each classical phase equation applies away from $t = 4$; the switch is handled by continuity, not by requiring a single derivative there.

Question 5

A series resistor-capacitor circuit has resistance $R = 1000 \, \Omega$ and capacitance $C = 0.001 \, \text{F}$. Its capacitor voltage is $U(t)$, charge is $q = CU$, and current is $I = q'$. The source voltage $E(t)$ obeys the loop relation $E = RI + U$. Initially the capacitor is uncharged. Time is in seconds.

A designer wants the exact voltage trajectory

$$U_d(t) = 6(1 - e^{-t/2}) \, \text{V}, \quad t \geq 0,$$

but the available source must satisfy $0 \leq E(t) \leq 5 \, \text{V}$. Ideal circuit laws are assumed; capacitor energy is $W = \frac{1}{2}CU^2$.

Tasks

1. Derive a first-order IVP for the capacitor voltage and identify the time constant, with units.
2. Find the unique source voltage and the current required to produce the desired trajectory exactly.
3. Determine the longest interval starting at 0 on which this trajectory respects the source bound. Find the capacitor voltage at the limiting time.
4. Calculate the stored energy at that time. Explain why simply fixing the source at 5 V afterward cannot continue the same prescribed trajectory.

Question 5 – Solution

Strategy. Use the circuit balance backward: a specified capacitor trajectory determines its current and therefore the required source.

Step 1: Derive the voltage model. Since $I = CU'$, the loop equation gives

$$RCU' + U = E(t), \quad U(0) = 0.$$

Here $RC = (1000)(0.001) = 1$ s is the time constant. With time numerically measured in seconds, the model is $U' + U = E$; the derivative coefficient carries the one-second time constant.

Step 2: Find the required input and current. The desired derivative is $U'_d = 3e^{-t/2}$ V/s, so

$$E(t) = 6 - 3e^{-t/2} \text{ V}, \quad I(t) = 0.003e^{-t/2} \text{ A}.$$

Substitution gives $RCU'_d + U_d = E$ and $U_d(0) = 0$. The prescribed U_d fixes its derivative, so the loop relation leaves no other possible source trajectory. In particular, the initial current is 3 mA and the required initial source is 3 V.

Step 3: Apply the source constraint. The required source increases strictly from 3 V toward 6 V. It first reaches 5 V when $e^{-t/2} = 1/3$. Therefore the full feasible initial interval is

$$0 \leq t \leq t_* = 2 \ln 3 \approx 2.197 \text{ s}, \quad U_d(t_*) = 4 \text{ V}.$$

Equality at the limiting time is allowed by the given source bound. Beyond it the unique required source exceeds 5 V.

Step 4: Compute energy and interpret saturation. At the limiting time,

$$W = \frac{1}{2}(0.001)(4)^2 = 0.008 \text{ J}.$$

A source fixed at 5 V afterward would instead produce $U = 5 - e^{-(t-t_*)}$ V, starting from 4 V. It tends to 5 V, whereas the desired trajectory tends to 6 V. Thus the circuit can continue operating, but it cannot continue this exact target under the source constraint. The feasible interval is a hardware limitation, not a singularity of the ideal differential equation.

Question 6

An inverted conical tank has height 4 m and top radius 2 m. It initially contains water to the brim. Let $h(t)$ be water depth above the outlet, with t in seconds. Assume horizontal water surfaces and similar conical cross-sections. The outflow rate is prescribed by

$$Q(h) = \kappa\sqrt{h}, \quad \kappa = 0.05 \text{ m}^{5/2}/\text{s},$$

while $h > 0$. There is no inflow. Ignore changes in the discharge coefficient and stop outflow once the tank is empty.

Tasks

1. Derive the volume as a function of depth and use volume conservation to form an IVP for h .
2. Solve for the depth until emptying and find the emptying time.
3. Find the fractions of the emptying time needed to reach half the original depth and half the original volume. Explain why these are different.
4. Sketch the physical depth, including the empty state. Decide whether depth and volume have continuously differentiable extensions through emptying, and explain the scope of the divided depth equation.

Question 6 – Solution

Strategy. The water surface area changes with depth. Convert a volume-loss law into a depth equation before separating variables.

Step 1: Use the conical geometry. Similarity gives $r = h/2$, so $V = \pi h^3/12$ and

$$\frac{\pi h^2}{4} h' = -0.05\sqrt{h}, \quad h' = -\frac{0.20}{\pi} h^{-3/2}, \quad h(0) = 4.$$

The first equality has units m^3/s . Division by h^2 is valid only while water remains.

Step 2: Integrate to emptying. Integrating $h^{3/2} dh = -(0.20/\pi) dt$ gives $h^{5/2} = 32 - t/(2\pi)$, hence

$$h(t) = (32 - t/(2\pi))^{2/5}, \quad 0 \leq t < t_e, \quad t_e = 64\pi \approx 201.06 \text{ s}.$$

Equivalently, $h = 4(1 - t/t_e)^{2/5}$. Differentiation verifies the depth equation; the physical depth is set to zero for $t \geq t_e$.

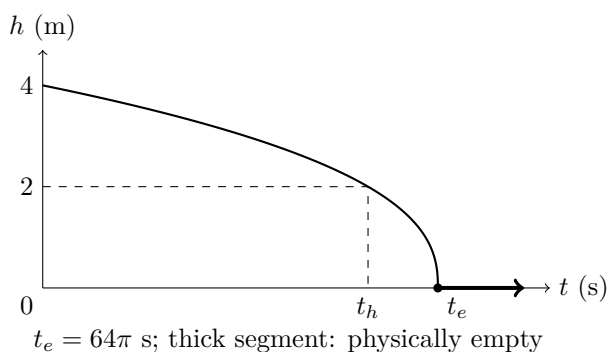
Step 3: Compare depth and volume targets. At $h = 2 \text{ m}$, $h/4 = 1/2$, giving

$$t_h/t_e = 1 - 2^{-5/2} \approx 0.8232.$$

At half volume, $(h/4)^3 = 1/2$, so

$$t_V/t_e = 1 - 2^{-5/6} \approx 0.4388.$$

Half depth leaves only one eighth of the original volume, so it takes substantially longer than losing half the volume.



Step 4: Interpret the endpoint. As $t \uparrow t_e$, $h' \rightarrow -\infty$, so the zero continuation of depth is continuous but not differentiable at emptying. However, $V' = -0.05\sqrt{h} \rightarrow 0$, agreeing with $V' = 0$ afterward; volume has a continuously differentiable zero continuation. The divided depth equation is a wet-tank model and is undefined at $h = 0$. It must not be continued algebraically into a fictitious refilling branch.

Question 7

A well-mixed pond maintains volume 1000 m^3 . Water enters and leaves at $100 \text{ m}^3/\text{day}$. The entering water has a constant pollutant concentration $u \geq 0 \text{ kg/m}^3$, which is to be selected. Pollutant also disappears inside the pond at a rate equal to 0.10 times its current mass per day. Initially the pond concentration is 0.08 kg/m^3 .

For this hypothetical design problem, require the pond concentration to stay at or below 0.10 kg/m^3 throughout the first five days. Assume perfect mixing and that disappearance does not change the liquid volume.

Tasks

1. Derive the pollutant mass balance and the concentration IVP, keeping flushing and internal disappearance distinct.
2. Solve for the concentration in terms of u and verify the initial condition.
3. Determine the largest constant incoming concentration satisfying the five-day requirement, and justify checking the entire time interval.
4. Find the largest constant incoming concentration that would satisfy the same bound for all future time. Explain why it differs from the finite-horizon answer.

Question 7 – Solution

Strategy. Separate input, outflow and internal loss, then use the solution's monotonicity to turn a whole-interval constraint into a justified endpoint test.

Step 1: Write the balances. If M is pollutant mass in kg and $c = M/1000$, then

$$M' = 100u - 100\frac{M}{1000} - 0.10M, \quad M(0) = 80.$$

Each term is in kg/day. Dividing by the constant volume gives

$$c' = 0.10u - 0.20c, \quad c(0) = 0.08.$$

The total loss coefficient combines flushing at 0.10 day^{-1} with internal disappearance at the same rate.

Step 2: Solve and identify monotonicity. An integrating factor $e^{0.20t}$ gives

$$c(t) = \frac{u}{2} + (0.08 - u/2)e^{-0.20t}, \quad t \geq 0.$$

Its initial value is 0.08, and differentiating verifies $c' = 0.10u - 0.20c$. It is decreasing when $u < 0.16$, constant when $u = 0.16$, and increasing when $u > 0.16$.

Step 3: Enforce the five-day bound. For $u \leq 0.16$, the maximum is the acceptable initial concentration. For $u > 0.16$, the maximum on $[0, 5]$ is $c(5)$. Therefore the largest allowed input is

$$u_5 = \frac{2(0.10 - 0.08e^{-1})}{1 - e^{-1}} \approx 0.2233 \text{ kg/m}^3.$$

This value exceeds 0.16, so its concentration rises monotonically from 0.08 to exactly 0.10. Any larger u fails at day five, proving optimality rather than just feasibility.

Step 4: Compare a permanent operating requirement. The limiting pond concentration is $u/2$. A bound valid for all $t \geq 0$ requires and is guaranteed by

$$u \leq 0.20 \text{ kg/m}^3.$$

For these inputs, $c(t)$ lies between its initial value and $u/2$. If $u > 0.20$, the limit exceeds the bound, so it must eventually be violated. The larger finite-horizon input uses the initially cleaner pond as temporary dilution capacity; it cannot be continued indefinitely under the same requirement.

Question 8

An isolated laboratory population is modeled by a per-capita growth rate that decreases linearly with population size:

$$P' = P(a - bP), \quad a > 0, b > 0.$$

Treat P as a continuous number of individuals, with time in days. Under unchanged laboratory conditions, the instantaneous total growth rate is 20 individuals/day both when $P = 100$ and when $P = 200$. A new culture begins with $P(0) = 100$.

Tasks

1. Identify a and b , including units, from the two instantaneous rate observations.
2. Solve the new culture's IVP and verify the result. State the assumptions supporting use of the fitted parameters in this culture.
3. Find when the population first reaches 250 individuals and explain why this is a unique future time.
4. An exponential model is fitted only to the initial population and its initial growth rate. Compare its rate prediction at $P = 200$ with the second observation, and explain what that discrepancy says about the model.

Question 8 – Solution

Strategy. Convert total growth observations to per-capita rates before estimating the crowding coefficient.

Step 1: Calibrate the rate law. The observations give

$$a - 100b = 0.20, \quad a - 200b = 0.10.$$

Subtracting yields

$$a = 0.30 \text{ day}^{-1}, \quad b = 0.001 \text{ individual}^{-1}\text{day}^{-1}.$$

The model becomes $P' = 0.30P(1 - P/300)$. Both specified populations give total rate 20 individuals/day.

Step 2: Solve the IVP. Separation for the branch $0 < P < 300$ gives

$$\ln \frac{P}{300 - P} = 0.30t + C.$$

Using $P(0) = 100$ gives $C = -\ln 2$ and

$$P(t) = \frac{300}{1 + 2e^{-0.30t}}, \quad t \geq 0.$$

If $z = 2e^{-0.30t}$, then $P' = 90z/(1 + z)^2 = 0.30P(1 - P/300)$, and the initial value is 100. Applying the calibration assumes the same fixed resources and crowding law, no migration, and a population scale for which the continuous approximation is appropriate.

Step 3: Locate the target. Setting $P = 250$ yields $e^{-0.30t} = 1/10$, so

$$t = \frac{\ln 10}{0.30} \approx 7.675 \text{ days}.$$

The formula increases strictly from 100 toward 300 for $t \geq 0$, so 250 is attained exactly once. Its denominator remains positive throughout this physical time range.

Step 4: Test the competing exponential model. For $P' = rP$, the initial observation forces $r = 20/100 = 0.20 \text{ day}^{-1}$. At $P = 200$, that model predicts 40 individuals/day, double the observed rate. Matching one population and one instantaneous slope does not validate a constant per-capita growth law.

The second observation distinguishes the two proposed laws. It supports the calibrated crowding model over this exponential alternative under the stated exact-data assumptions, but two rate observations alone do not establish that the model remains accurate at every population size.

Question 9

A spherical droplet has constant density ρ and radius $r(t)$. Assume its mass disappears at a rate proportional to its current surface area, with a constant coefficient $k > 0$:

$$M = \frac{4}{3}\pi\rho r^3, \quad M' = -k(4\pi r^2)$$

while the droplet exists. Radius decreases from 3 mm to 2 mm during the first 5 minutes. Ignore changes in shape, density and environmental conditions.

Tasks

1. Derive the radius IVP from the mass balance and identify k/ρ and its units from the observations. Can k itself be found without knowing ρ ?
2. Find the extinction time and the fraction of original mass remaining at any time before extinction.
3. Determine when half the original mass remains; compare this with the time at which the radius is halved.
4. Fit an exponential mass-loss model with the same initial mass and initial mass-loss rate. Compare the two predictions at 10 minutes and discuss their extinction predictions.

Question 9 – Solution

Strategy. Differentiate the geometric mass formula before applying the surface-loss law. A constant radius-loss rate need not imply exponential mass loss.

Step 1: Reduce the balance to radius. For $r > 0$, the chain rule gives

$$4\pi\rho r^2 r' = -4\pi k r^2, \quad r' = -k/\rho, \quad r(0) = 3 \text{ mm}.$$

The observed radius loss is 1 mm in 5 min, hence

$$\boxed{k/\rho = 0.20 \text{ mm/min}}, \quad \boxed{r(t) = 3 - 0.20t \text{ mm}}.$$

If density is in mass/mm^3 , k has units $\text{mass}/(\text{mm}^2 \text{ min})$. The observations identify only the ratio; a density value is needed to determine k itself.

Step 2: Determine mass and extinction. The radius reaches zero at $\boxed{t_e = 15 \text{ min}}$. Since mass is proportional to r^3 ,

$$\boxed{\frac{M(t)}{M(0)} = (1 - t/15)^3}, \quad 0 \leq t \leq 15.$$

The physical mass and radius remain zero afterward. Extending the linear radius formula past 15 would produce negative radius and is not part of the droplet model. Differentiating the cubic mass formula recovers the original surface-loss balance while $r > 0$.

Step 3: Compare fractional targets. Half mass occurs when $(1 - t/15)^3 = 1/2$, giving

$$\boxed{t_{1/2,M} = 15(1 - 2^{-1/3}) \approx 3.094 \text{ min}}.$$

Half radius occurs at $t = 7.5 \text{ min}$, when only $(1/2)^3 = 1/8$ of the initial mass remains. Geometric scaling makes these distinct events.

Step 4: Test an exponential alternative. The initial relative mass slope from the cubic formula is $M'(0)/M(0) = -1/5 \text{ min}^{-1}$. The matching exponential is therefore

$$M_{\text{exp}}(t) = M(0)e^{-t/5}.$$

At 10 min the surface-loss model predicts $M/M(0) = 1/27 \approx 0.0370$, whereas the exponential predicts $e^{-2} \approx 0.1353$. The exponential never reaches zero at finite time; the geometric model empties at 15 min. Agreement of initial value and slope does not imply agreement of the physical loss mechanism or its later predictions.

Question 10

A reservoir initially contains 10 L of usable liquid. During a four-minute operation, a pump supplies a constant rate $u \geq 0$ L/min, scheduled demand removes liquid at rate $d(t) = 2 + t$ L/min, and leakage removes liquid at a rate equal to 0.50 times the current volume per minute. Here t is in minutes; the slope in $d(t)$ is 1 L/min².

The reservoir must retain at least 4 L at every time $0 \leq t \leq 4$. Assume no capacity restriction, no other flows, and that the stated demand and leakage laws apply while liquid remains.

Tasks

1. Derive and solve the volume IVP in terms of u .
2. Find the smallest constant pump rate that satisfies the reserve requirement throughout the operation. Justify feasibility over the whole interval, not only at its endpoint.
3. For this rate, calculate the cumulative leaked volume by $t = 4$ using an overall volume balance.
4. Compare the answer with the minimum pump rate obtained by ignoring leakage. Sketch the optimal volume and reserve level, and explain why the no-leakage design cannot meet the original requirement.

Question 10 – Solution

Strategy. Solve the storage balance, obtain a necessary pump rate from the final reserve, and then verify that this candidate satisfies the full time constraint.

Step 1: Model and solve the storage balance. Input minus demand minus leakage gives

$$S' = u - (2 + t) - 0.50S, \quad S(0) = 10,$$

with all rates in L/min. A particular solution is $2u - 2t$; adding $Ae^{-t/2}$ and applying $S(0) = 10$ gives

$$S(t) = 2u - 2t + (10 - 2u)e^{-t/2}.$$

It has initial value 10, and differentiation gives the required balance on any time interval on which the physical operation remains valid.

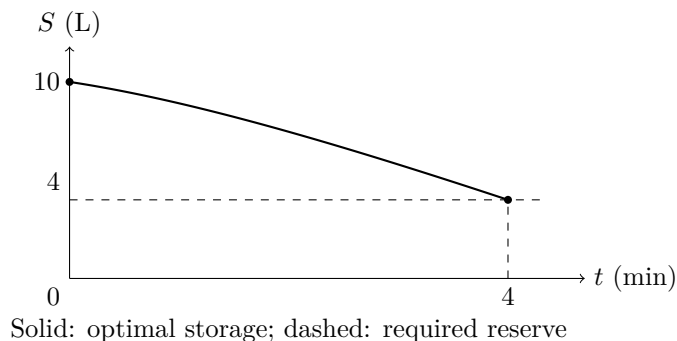
Step 2: Find and verify the minimum rate. Since $S(4) \geq 4$ is necessary,

$$u \geq u_* = \frac{12 - 10e^{-2}}{2(1 - e^{-2})} = 6 + \frac{1}{e^2 - 1} \approx 6.157 \text{ L/min}.$$

For this candidate, $6 < u_* < 7$ and

$$S'(t) = -2 + (u_* - 5)e^{-t/2} < 0 \quad (0 \leq t \leq 4).$$

Thus S decreases from 10 to exactly 4 L and never violates the reserve. The candidate is feasible, and every smaller rate fails the necessary final condition, proving optimality.



Step 3: Recover cumulative leakage. The scheduled demand totals $\int_0^4 (2 + t) dt = 16$ L. Initial storage plus pumped volume equals final storage plus demand plus leakage, so

$$L_{\text{leak}} = 10 + 4u_* - 16 - 4 = 4u_* - 10 \approx 14.63 \text{ L}.$$

Step 4: Compare the incomplete model. Ignoring leakage gives $S_0 = 10 + (u - 2)t - t^2/2$. This concave function has its minimum on $[0, 4]$ at an endpoint; $S_0(4) \geq 4$ gives the no-leakage minimum $u = 2.5$ L/min. That rate is below u_* and fails the actual reserve constraint. It may eventually exhaust the actual reservoir, so negative values of its unconstrained storage formula would not describe physical stored liquid. The optimal design above stays positive and avoids that limitation.

Question 1

Consider the autonomous equation

$$y' = y(1 - y)(y - 3), \quad y(0) = a \in \mathbb{R}.$$

An equilibrium is a constant solution. Call it stable if sufficiently small initial perturbations stay as small as prescribed for all future time, and asymptotically stable if it is also locally attracting. A basin of attraction is the set of initial values whose forward solutions exist for all $t \geq 0$ and tend to that equilibrium.

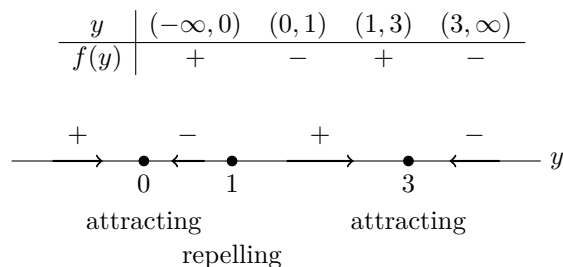
Tasks

1. Find all equilibria and construct a phase line, showing the sign of y' on every interval between them.
2. Classify the equilibria as asymptotically stable or unstable, explaining the role of the arrows.
3. Determine the forward limit for every real initial value a and the exact basins of the attracting equilibria.
4. Justify global forward existence and explain why a nonconstant solution cannot cross an equilibrium. Do not rely on an explicit solution formula.

Question 1 – Solution

Strategy. Use sign intervals and uniqueness to trap solutions; bounded monotone trajectories reveal their limits.

Step 1: Find the zeros and arrows. The only zeros of $f(y) = y(1 - y)(y - 3)$ are 0, 1, 3. Its signs are



Step 2: Classify the equilibria. Arrows point toward 0 and 3 on both sides, so both are asymptotically stable. The arrows on either side of 1 point away, so 1 is unstable. Nearby solutions remain between their initial value and the attracting zero, which supplies stability as well as attraction.

Step 3: Determine every forward limit. The classification is

$$\lim_{t \rightarrow \infty} y(t) = \begin{cases} 0, & a < 1, \\ 1, & a = 1, \\ 3, & a > 1. \end{cases}$$

In particular, the basins are $(-\infty, 1)$ for 0 and $(1, \infty)$ for 3. These include the attracting equilibria themselves. The value $a = 1$ stays exactly at the unstable equilibrium.

Step 4: Justify the global conclusions. Since f is a polynomial, local solutions are unique. A nonconstant solution cannot meet an equilibrium at a finite time: local uniqueness there, in both time directions, would force the two solutions to agree.

Every trajectory is trapped in a bounded interval between its initial value and adjacent equilibria. A smooth right-hand side permits continuation while the solution remains bounded, so all forward solutions are global. Each nonconstant trajectory is monotone and bounded. If its limit L had $f(L) \neq 0$, continuity would keep its speed bounded away from zero near L , contradicting convergence. The accessible zero in its direction of motion is therefore its limit.

Question 2

Consider

$$y' = -(y + 1)(y - 2)^2.$$

Call an equilibrium semistable if it attracts from one side and repels from the other; such an equilibrium is not stable under unrestricted perturbations from both sides.

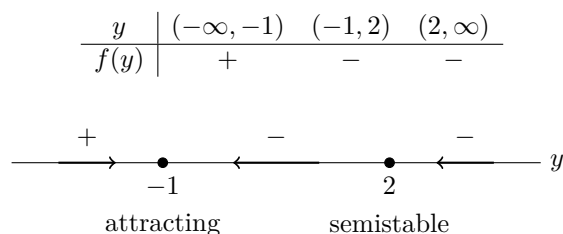
Tasks

1. Find the equilibria and construct a complete phase line, including signs on both sides of a repeated zero.
2. Classify each equilibrium and specify the attracting side of any semistable equilibrium.
3. Find the forward limits for initial values $y(0) = -2$, 1 , and 3 , with a justification of forward existence.
4. Evaluate f' at each equilibrium. Explain why a zero derivative is not itself a stability classification and why the repeated factor does not reverse the sign of f .

Question 2 – Solution

Strategy. Preserve the sign across a squared factor; use one-sided attraction and repulsion when the derivative test is inconclusive.

Step 1: Build the sign chart. The equilibria are -1 and 2 . Away from them, $(y - 2)^2 > 0$, so the sign of f is the opposite of the sign of $y + 1$:



Step 2: Classify with the sides stated. The equilibrium -1 is asymptotically stable. At 2 , solutions above it decrease toward it, while those just below decrease away toward -1 . Hence

2 is semistable: attracting from above, repelling from below.

It is unstable in the two-sided stability sense: however close an initial value is below 2 , its trajectory eventually leaves, for example, the interval $(1, 3)$.

Step 3: Track the three initial values. Uniqueness prevents crossing either equilibrium. The solution starting at -2 increases toward -1 ; that starting at 1 decreases toward -1 ; that starting at 3 decreases toward 2 . Each remains in a bounded closed interval, so the smooth equation has a global forward solution in each case. Monotone convergence and the nonzero-speed contradiction identify the stated limits.

Step 4: Interpret the derivative test. Differentiating gives $f'(y) = -(y - 2)^2 - 2(y + 1)(y - 2)$, so

$$f'(-1) = -9 < 0, \quad f'(2) = 0.$$

The negative derivative confirms attraction at -1 . At 2 , the linearized equation contains no nonzero first-order term and makes no decision. The sign chart supplies the missing information. A squared factor is positive on both sides of its zero, so it touches zero without a sign reversal.

Question 3

Two proposed relaxation laws are

$$\text{A: } y' = -y^3, \quad \text{B: } y' = y^3.$$

Both have an equilibrium at 0 and derivative of the right-hand side equal to zero there.

Tasks

1. Classify 0 for each law using the sign of the right-hand side. Explain why their identical derivative-test data do not imply identical stability.
2. Solve both IVPs with $y(0) = a$, including $a = 0$, and specify their maximal forward time ranges.
3. For law A, prove stability directly from the solution and determine its large-time decay for $a \neq 0$.
4. For law B, use the solution to prove instability and find the blow-up time for $a \neq 0$. Distinguish leaving a small neighborhood from blowing up.

Question 3 – Solution

Strategy. When the linear term vanishes, retain the nonlinear term. Separation gives both a stability proof and the actual time scales.

Step 1: Classify from signs. For A, $-y^3$ is positive below 0 and negative above, so trajectories move toward 0. For B the signs reverse, so trajectories move away. Thus 0 is asymptotically stable for A and unstable for B. The shared value $f'(0) = 0$ is an inconclusive test, not evidence of neutral stability.

Step 2: Solve with the initial sign preserved. For $a \neq 0$, differentiating y^{-2} gives 2 for A and -2 for B. Applying the initial value yields

$$y_A(t) = \frac{a}{\sqrt{1+2a^2t}}, \quad t \geq 0,$$

$$y_B(t) = \frac{a}{\sqrt{1-2a^2t}}, \quad 0 \leq t < \frac{1}{2a^2}.$$

The full maximal intervals containing 0 are $(-1/(2a^2), \infty)$ and $(-\infty, 1/(2a^2))$, respectively. Direct differentiation verifies the equations. If $a = 0$, the unique solution of either smooth equation is zero for all real time.

Step 3: Prove stability and attraction for A. For $t \geq 0$, $|y_A(t)| \leq |a|$. Given $\varepsilon > 0$, choosing $\delta = \varepsilon$ ensures $|a| < \delta$ implies $|y_A(t)| < \varepsilon$ for all future time. Also $y_A(t) \rightarrow 0$ for every a , so the attraction is global. More precisely,

$$\sqrt{2t}|y_A(t)| \rightarrow 1 \quad (a \neq 0).$$

The decay is algebraic rather than exponential.

Step 4: Prove instability for B. Choose any fixed $\varepsilon > 0$ and any nonzero $|a| < \varepsilon$. The magnitude first reaches ε at

$$t_\varepsilon = \frac{1}{2a^2} - \frac{1}{2\varepsilon^2} > 0,$$

then exceeds it. Arbitrarily small nonzero perturbations therefore leave that neighborhood. Blow-up occurs later, at $1/(2a^2)$, with the sign of a . Instability only requires departure from a prescribed neighborhood; it does not require an infinite solution value.

Question 4

Construct a real polynomial autonomous equation $y' = f(y)$ with exactly the equilibria $-2, 0, 3$ and these properties:

- -2 is asymptotically stable;
- 0 attracts from above and repels from below;
- 3 is unstable;
- $f(1) = -6$.

Tasks

1. Determine the smallest possible degree, using the sign changes required at the three zeros.
2. Construct the polynomial of that degree and prove uniqueness among polynomials of that degree satisfying all the requirements.
3. Draw its phase line and verify each requested stability property.
4. Find the forward limits for $y(0) = -1$ and $y(0) = 1$. Explain why knowing the equilibrium locations alone would not determine these limits.

Question 4 – Solution

Strategy. Translate attraction and repulsion into signs, then into the parity of each root's multiplicity.

Step 1: Establish the minimum degree. At a two-sided attracting or repelling equilibrium, the sign must reverse, requiring an odd root multiplicity. At the specified semistable zero, the sign must be negative on both sides, requiring even multiplicity. The smallest multiplicities at $-2, 0, 3$ are therefore 1, 2, 1, so the degree is at least 4.

Step 2: Determine the coefficient. Every degree-four candidate must have the form

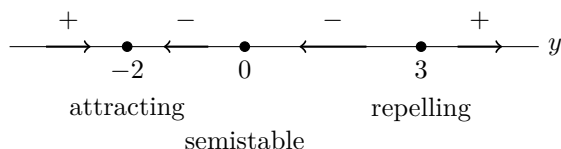
$$f(y) = K(y + 2)y^2(y - 3), \quad K \neq 0.$$

At $y = 1$ this gives $f(1) = -6K$. The prescribed rate forces $K = 1$, hence

$$y' = (y + 2)y^2(y - 3).$$

This has exactly the three requested real zeros. The forced multiplicities exhaust degree four, and the rate condition fixes the only remaining coefficient, proving uniqueness at the minimum degree.

Step 3: Verify the phase line. The signs are $+, -, -, +$ on the four successive intervals cut by $-2, 0, 3$.



Thus arrows approach -2 from both sides; near 0 they point left on both sides, attracting from above and repelling from below; and they point away from 3 on both sides. These are precisely the requested behaviors.

Step 4: Infer the selected limits. Starting at -1 , the solution decreases within $(-2, 0)$ and tends to $\boxed{-2}$. Starting at 1 , it decreases within $(0, 3)$ and tends to $\boxed{0}$. Smoothness and bounded trapping ensure global forward existence for both, and uniqueness prevents crossing the bounding equilibria.

Equilibrium locations only identify zeros. Changing root multiplicities or the overall sign can change the arrows and therefore the limiting behavior, even when the zero set is unchanged. The stability and rate requirements supply information that the locations alone do not.

Question 5

Consider the one-parameter family

$$y' = \mu - y^2, \quad \mu \in \mathbb{R}.$$

Tasks

1. Find and classify all equilibria for $\mu < 0$, $\mu = 0$, and $\mu > 0$. State the one-sided behavior at any semistable case.
2. Draw the equilibrium branches in the (μ, y) -plane, distinguishing attracting and repelling branches and marking the exceptional parameter value.
3. Solve the IVP $y(0) = 0$ for each parameter regime and determine its forward behavior and maximal forward time range.
4. Explain how the number and stability of equilibria change at $\mu = 0$, and why the derivative test alone does not settle that parameter value.

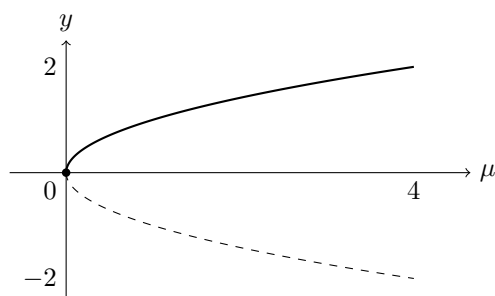
Question 5 – Solution

Strategy. Compare a horizontal parameter with y^2 , and treat the double zero separately from the two simple roots.

Step 1: Classify the parameter regimes. If $\mu < 0$, there are no equilibria and $y' < 0$ everywhere. If $\mu = 0$, the only equilibrium is 0: $-y^2 < 0$ on both sides, so it attracts from above and repels from below. If $\mu > 0$, the equilibria are

$$y = -\sqrt{\mu} \text{ (unstable),} \quad y = \sqrt{\mu} \text{ (asymptotically stable)}.$$

Indeed $\mu - y^2$ is positive between the roots and negative outside them. The derivative $f_y = -2y$ confirms the classifications at the simple roots.



Solid: attracting; dashed: repelling; origin: semistable

Step 2: Solve the selected IVP. Separation and $y(0) = 0$ give the following cases, with $a > 0$ as indicated:

parameter	$y(t)$	forward time range
$\mu > 0, a = \sqrt{\mu}$	$a(1 - e^{-2at})/(1 + e^{-2at})$	$0 \leq t < \infty$
$\mu = 0$	0	$0 \leq t < \infty$
$\mu < 0, a = \sqrt{-\mu}$	$-a \tan(at)$	$0 \leq t < \pi/(2a)$

For $\mu > 0$, integration on $-a < y < a$ gives $\ln((a + y)/(a - y)) = 2at$. For the last case, integration gives $\arctan(y/a) = -at$ on the initial branch. Differentiating the displayed expressions verifies $y' = \mu - y^2$ and their zero initial values.

Step 3: Describe the future. For $\mu > 0$, the selected solution increases to $\sqrt{\mu}$; at $\mu = 0$ it stays at zero; for $\mu < 0$ it decreases to $-\infty$ as $t \uparrow \pi/(2a)$, so no finite continuous extension exists.

Step 4: Interpret the branch merger. As μ decreases to zero, the attracting and repelling equilibria merge into a semistable double zero; for negative μ neither remains. At the merger $f_y(0) = 0$, so the derivative test is inconclusive. The sign of $-y^2$ supplies the one-sided classification.

Question 6

For a fixed real constant L , compare the two equations on $t \geq 0$:

$$\text{A: } y' = -\frac{y-L}{(1+t)^2}, \quad \text{B: } y' = -\frac{y-L}{1+t}.$$

Their right-hand sides depend on time. Nevertheless, at every finite time each points toward L whenever $y \neq L$. Use stability relative to the initial time $t = 0$.

Tasks

1. Find every constant solution of each equation and explain why these are equilibria despite the explicit time dependence.
2. Solve both IVPs with $y(0) = a$ and verify the initial values and differential equations.
3. Decide whether L is stable and whether it is asymptotically stable for each equation. Prove the claims directly from the solutions.
4. Explain why arrows pointing toward L at every finite time do not guarantee attraction in both models. Relate the distinction to the accumulated coefficient integral.

Question 6 – Solution

Strategy. Measure the displacement from L . Its reduction depends on the total accumulated damping, not just the sign at each instant.

Step 1: Identify the constant solutions. A constant $y = c$ must make the right-hand side zero for every $t \geq 0$. Both coefficients are strictly positive, so the only choice in either equation is $\boxed{c = L}$. Explicit time dependence does not preclude a constant solution.

Step 2: Solve the displacement equations. Let $z = y - L$, so $z(0) = a - L$. For A,

$$\int_0^t \frac{ds}{(1+s)^2} = 1 - \frac{1}{1+t} = \frac{t}{1+t}, \quad \boxed{y_A = L + (a - L)e^{-t/(1+t)}}.$$

For B, the corresponding integral is $\ln(1+t)$, giving

$$\boxed{y_B = L + \frac{a - L}{1+t}}.$$

Both formulas have value a at zero. Their displacement derivatives are exactly the prescribed negative coefficient times their displacement, verifying the equations. They exist for all $t \geq 0$.

Step 3: Separate stability from attraction. For both models, $|y(t) - L| \leq |a - L|$. Given $\varepsilon > 0$, $\delta = \varepsilon$ proves stability at initial time zero. But

$$\lim_{t \rightarrow \infty} y_A = L + (a - L)e^{-1}, \quad \lim_{t \rightarrow \infty} y_B = L.$$

Thus L is **stable but not asymptotically stable for A**; no other initial value converges to L . For B, L is **globally asymptotically stable**.

Step 4: Explain the limitation of the arrow argument. In a scalar equation $z' = -b(t)z$, the solution is $z(0) \exp(-\int_0^t b(s) ds)$. A has positive but integrable damping, with total integral 1; a fixed fraction of any nonzero initial displacement remains. B has divergent accumulated damping and removes the displacement completely.

For a continuous autonomous equation, a non-equilibrium limiting value would have nonzero limiting speed. Here the coefficient can tend to zero with time, so that autonomous argument does not apply. Instantaneous inward arrows alone are insufficient for a time-dependent equation.

Question 7

Consider the continuous, non-Lipschitz equation

$$y' = 2\sqrt{|y|}, \quad y(0) = 0, \quad t \geq 0.$$

When solutions are not unique, call an equilibrium stable only if every forward solution from every sufficiently close initial value remains within any prescribed neighborhood.

Tasks

1. Find all equilibria and determine the sign of the right-hand side away from them.
2. For each $c \geq 0$, construct a solution that stays at zero through time c and then becomes positive. Verify differentiability at the joining time.
3. Construct a continuously differentiable solution with $y(0) = -1$ that crosses the equilibrium level at $t = 1$, and sketch it.
4. Prove that the equilibrium is unstable under the stated convention. Explain precisely why the usual prohibition against crossing an equilibrium does not apply.

Question 7 – Solution

Strategy. Verify explicit joined solutions instead of invoking a uniqueness theorem whose hypothesis fails.

Step 1: Find the zero and signs. The only equilibrium is $y = 0$. The right-hand side is positive for every $y \neq 0$, so motion is upward on either side. Below zero the direction is toward the equilibrium level, but arrival need not force the solution to stay there.

Step 2: Verify delayed departures. For any $c \geq 0$, define

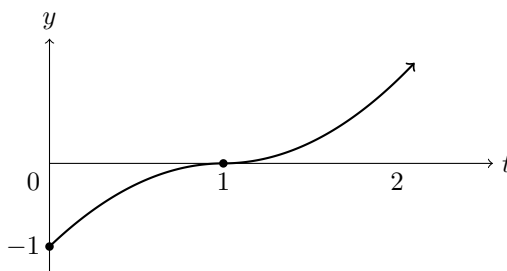
$$y_c(t) = \begin{cases} 0, & 0 \leq t \leq c, \\ (t - c)^2, & t \geq c. \end{cases}$$

Before c both sides of the equation vanish; afterward $y'_c = 2(t - c) = 2\sqrt{|y_c|}$. At c , both values and both one-sided derivatives are zero. The difference quotient also gives derivative zero, so the joined solution is continuously differentiable. The identically zero function is another solution.

Step 3: Cross from below. A crossing solution is

$$y(t) = \begin{cases} -(1 - t)^2, & 0 \leq t \leq 1, \\ (t - 1)^2, & t \geq 1. \end{cases}$$

Before 1, $y' = 2(1 - t) = 2\sqrt{|y|}$; afterward the same check as above applies. Both derivatives at the join are zero, and $y(0) = -1$.



A classical solution crosses $y = 0$ at $t = 1$

Step 4: Diagnose instability and failed uniqueness. For any $\delta > 0$, initial value 0 is within δ of the equilibrium, yet the solution $y_0(t) = t^2$ leaves the fixed neighborhood $(-1, 1)$ after $t = 1$. This violates the stated requirement for every forward solution, so the equilibrium is unstable.

Local Lipschitz regularity fails since $|f(y) - f(0)|/|y| = 2/\sqrt{|y|}$ is unbounded near zero. The crossing and constant solutions agree at $(1, 0)$ but have different continuations. The usual noncrossing proof requires local uniqueness; continuity alone does not provide it.

Question 8

Consider the autonomous equation with its stated domain:

$$y' = \frac{y-1}{y+2}, \quad y \neq -2.$$

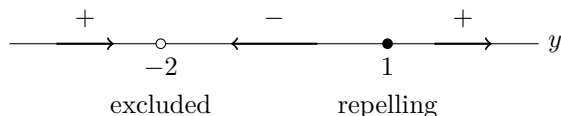
Tasks

1. Identify all equilibria and construct a phase line that distinguishes an excluded level from a zero of the right-hand side.
2. Classify each equilibrium and determine the direction of motion in every component cut by the equilibrium and the excluded level.
3. For $y(0) = 0$, derive an implicit solution relation and find the exact maximal forward time range.
4. Determine the limiting value and slope at its finite endpoint. Explain why approaching a finite level does not make that level an attracting equilibrium and why multiplying by $y + 2$ does not remove the obstruction for this original equation.

Question 8 – Solution

Strategy. A denominator zero splits the domain but is not an equilibrium. Test whether the selected trajectory reaches that boundary in finite time.

Step 1: Separate zeros from exclusions. The only equilibrium is $y = 1$. The value $y = -2$ is outside the equation's domain. The signs of f are positive on $(-\infty, -2)$, negative on $(-2, 1)$, and positive on $(1, \infty)$.



Thus 1 is unstable: arrows point away on both sides. Also $f'(y) = 3/(y+2)^2$, so $f'(1) = 1/3 > 0$ confirms the classification.

Step 2: Integrate the selected branch. Starting at 0, the solution decreases within $(-2, 1)$. Separation gives

$$dt = \frac{y+2}{y-1} dy, \quad t = y + 3 \ln(1-y).$$

The constant is zero from $y(0) = 0$. On $-2 < y \leq 0$, the derivative of the right side with respect to y is $(y+2)/(y-1) < 0$. It is therefore invertible along this branch, and implicit differentiation verifies the original equation.

Step 3: Determine the endpoint. As $y \downarrow -2$, the time tends to

$$T = -2 + 3 \ln 3 \approx 1.296, \quad 0 \leq t < T.$$

This is the maximal forward range: the inverse exists until T , but at T its limiting value is excluded. Within the branch,

$$y(t) \rightarrow -2^+, \quad y'(t) = \frac{y-1}{y+2} \rightarrow -\infty.$$

Step 4: Interpret the domain boundary. An attracting equilibrium must itself define a constant solution of the given equation. Here -2 does not, and the selected solution reaches the boundary in finite time with unbounded slope rather than converging to a valid equilibrium as $t \rightarrow \infty$.

Multiplication gives $(y+2)y' = y-1$, equivalent to the original only when $y \neq -2$. At $y = -2$ it would demand $0 = -3$ for any finite derivative. Thus even the multiplied relation supplies no differentiable continuation through this boundary.

Question 9

Let

$$V(y) = \frac{1}{4}(y^2 - 1)^2, \quad y' = -V'(y).$$

Think of V as an energy function; no physical units are assumed.

Tasks

1. Derive the autonomous equation explicitly, find its equilibria, and classify them using the signs of the right-hand side.
2. Compute $dV(y(t))/dt$ along a solution. Identify exactly when this derivative vanishes.
3. Sketch V and relate its local minima and maximum to equilibrium stability. Explain why nonincreasing energy alone does not imply convergence to a global minimum for every initial value.
4. Determine the forward limit for every real initial value and justify global forward existence. State the basins of the attracting equilibria.

Question 9 – Solution

Strategy. Combine the energy identity with the phase line; energy decrease is informative but does not by itself choose a limiting equilibrium.

Step 1: Derive the flow and classify its zeros. Since $V'(y) = y(y^2 - 1)$,

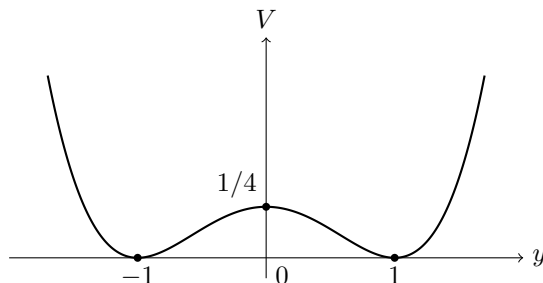
$$y' = y - y^3 = y(1 - y^2).$$

Its zeros are $-1, 0, 1$. Its signs on the successive intervals are $+, -, +, -$, so -1 and 1 are asymptotically stable and 0 is unstable.

Step 2: Differentiate the energy. The chain rule gives

$$\frac{dV}{dt} = V'(y)y' = -[V'(y)]^2 = -(y - y^3)^2 \leq 0.$$

The derivative vanishes exactly at $y = -1, 0, 1$. On any nonconstant trajectory it is strictly negative, since uniqueness prevents meeting an equilibrium at a finite time.



Energy minima: ± 1 ; local maximum: 0

Step 3: Interpret the energy landscape. The points ± 1 are minima with $V = 0$; 0 is a local maximum with $V = 1/4$. Since the flow is down the energy gradient, the minima attract and the maximum repels. Nevertheless, the initial value $y(0) = 0$ stays at that maximum forever, with constant energy. Nonincreasing energy does not force every solution to reach a global minimum, nor distinguish which minimum will be selected.

Step 4: Determine the basins and future limits. The decreasing-energy bound confines a trajectory to $\{y : V(y) \leq V(y(0))\}$, a bounded set because $V(y) \rightarrow \infty$ as $|y| \rightarrow \infty$. Smoothness then ensures global forward continuation. Phase-line trapping gives bounded monotonic motion for every nonconstant solution. Its limit must be a zero of $y - y^3$, and it cannot cross zero. Therefore

$$\lim_{t \rightarrow \infty} y(t) = \begin{cases} -1, & y(0) < 0, \\ 0, & y(0) = 0, \\ 1, & y(0) > 0. \end{cases}$$

The basins are $(-\infty, 0)$ for -1 and $(0, \infty)$ for 1 .

Question 10

Compare two autonomous equations on the real line:

$$\text{A: } y' = -y, \quad \text{B: } y' = -\frac{y}{1+y^2}.$$

For the quantitative comparison, use the same initial value $y(0) = a > 0$.

Tasks

1. Compare the equilibrium sets, phase-line arrows and derivatives of the right-hand sides at the equilibria. Prove the stability classification for both equations.
2. Find the solution of A and an implicit relation for B. Justify global forward existence and convergence for B.
3. Derive the time required to decrease from a to $a/2$ for each equation. Explain how the difference depends on a .
4. For $a = 2$, sketch both trajectories and determine which is larger for every $t > 0$. Explain what identical equilibrium and local derivative data fail to specify.

Question 10 – Solution

Strategy. Multiplication by a positive state-dependent factor preserves the direction of motion, but changes the time spent along the same phase-line path.

Step 1: Compare equilibrium information. Both equations have only the equilibrium 0, with positive velocity below zero and negative velocity above. Both right-hand-side derivatives at zero equal -1 . In each model, a solution is trapped between its initial value and zero, which proves stability. Smoothness, boundedness and monotonicity give global forward existence and convergence to the only accessible zero. Thus both equilibria are globally asymptotically stable.

Step 2: Solve the selected branches. For A, $y_A = ae^{-t}$. For B with $y > 0$, separation gives

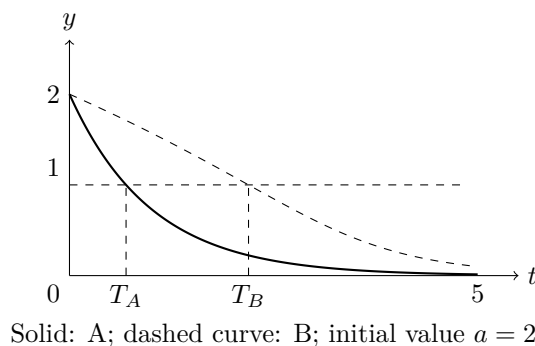
$$dt = -(1/y + y)dy, \quad t = \ln(a/y) + \frac{a^2 - y^2}{2}.$$

The right side decreases strictly with y on $(0, a]$, is zero at a , and tends to infinity as $y \downarrow 0$. It therefore defines a unique positive solution for every $t \geq 0$. Implicit differentiation recovers B and confirms that zero is approached only in infinite time.

Step 3: Compare halving times. Setting $y = a/2$ yields

$$T_A = \ln 2, \quad T_B = \ln 2 + \frac{3a^2}{8}.$$

The extra time in B grows quadratically with the initial amplitude. It tends to zero as $a \downarrow 0$, consistent with the identical local derivatives, but can be arbitrarily large for large a .



Step 4: Order the trajectories. For $t > 0$, the B solution satisfies $0 < y_B < a$, so its implicit equation gives $t > \ln(a/y_B)$. Hence $y_B > ae^{-t} = y_A$. The graph uses $a = 2$, for which $T_B = \ln 2 + 3/2$.

The factor $1/(1 + y^2)$ is positive and equals 1 at zero, preserving phase directions and the local linear rate. Away from zero it slows the motion. Equilibria and local stability data alone do not specify finite-amplitude travel times or entire solution curves.

Question 1

Consider

$$y' = t - y, \quad y(0) = 1.$$

For a step h , explicit Euler uses $t_{n+1} = t_n + h$ and $Y_{n+1} = Y_n + hf(t_n, Y_n)$, where Y_n approximates $y(t_n)$. Use exact arithmetic in intermediate steps unless rounding is explicitly requested.

Tasks

1. Starting at $t_0 = 0$, use $h = 1/4$ to compute every Euler value through $t = 1$. Include the slope used on each step.
2. Find the exact solution and the signed endpoint error $Y_4 - y(1)$, reporting its numerical value to six decimal places.
3. Draw the Euler polygon and the exact solution on $[0, 1]$. Explain what the slope of each straight segment represents.
4. A program obtains $Y_1 = 0.8125$ by evaluating $t - y$ at (t_1, Y_0) . Identify the error and show the correct first update. Is that program implementing the stated Euler formula?

Question 1 – Solution

Strategy. Evaluate each slope at the current numerical point, then move both time and the approximation to the next node.

Step 1: Compute the four updates. The recurrence is $Y_{n+1} = Y_n + (t_n - Y_n)/4$. Its exact values are

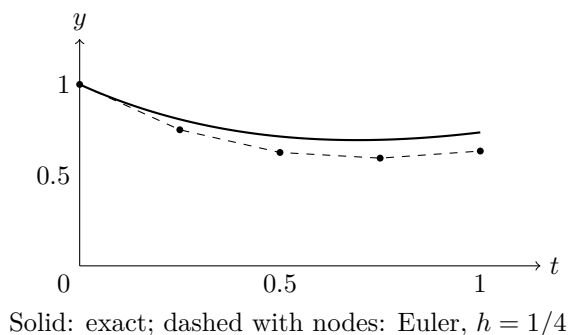
n	t_n	Y_n	$f(t_n, Y_n)$
0	0	1	-1
1	1/4	3/4	-1/2
2	1/2	5/8	-1/8
3	3/4	19/32	5/32
4	1	81/128	not needed

Thus $Y_4 = 81/128 = 0.6328125$.

Step 2: Check against the exact IVP. Solving $y' + y = t$ with integrating factor e^t gives

$$y = t - 1 + 2e^{-t}, \quad Y_4 - y(1) = \frac{81}{128} - \frac{2}{e} \approx -0.102946.$$

The exact formula has $y(0) = 1$ and derivative $1 - 2e^{-t} = t - y$, so it is the correct comparison solution.



Step 3: Interpret the polygon. On $[t_n, t_{n+1}]$, the Euler segment has slope $f(t_n, Y_n)$. It is tangent to the solution through the numerical point (t_n, Y_n) , which generally differs from the original exact solution after the first step. The polygon is continuous but usually has corners at its nodes.

Step 4: Diagnose the time index. The correct first update is $1 + (1/4)(0 - 1) = 0.75$. The program instead uses $1 + (1/4)(1/4 - 1) = 0.8125$, mixing the new time with the old state. It is not the stated explicit Euler method, which requires both arguments from the same current node.

Question 2

For $y' = y$, $y(0) = 1$, apply explicit Euler on $[0, 1]$ with N equal steps and $h = 1/N$. Let Y_N denote its endpoint value. You may use $\ln(1 + h) = h - h^2/2 + O(h^3)$ and $e^z = 1 + z + O(z^2)$ as the arguments tend to zero.

Tasks

1. Derive a closed formula for Y_N and prove that it underestimates the exact endpoint for every positive integer N .
2. Compute the one-step defect when Euler starts from an exact value $y(t_n)$. Show that this defect is of order h^2 .
3. Derive the leading endpoint error as $h \rightarrow 0$ and explain why its order differs from that of the one-step defect.
4. Calculate endpoint values and absolute errors for $N = 2, 4, 8$. Explain what error reduction is expected when N doubles and what the computation cost does.

Question 2 – Solution

Strategy. Solve the discrete recurrence exactly, then compare its logarithm with the continuous exponential growth.

Step 1: Find and bound the endpoint. Euler gives $Y_{n+1} = (1 + h)Y_n$, so

$$Y_N = (1 + 1/N)^N.$$

The exact solution is e^t . Since $\ln(1 + h) < h$ for $h > 0$, $N \ln(1 + 1/N) < 1$ and hence $Y_N < e$. This is an underestimate for every grid, not merely for the three grids below.

Step 2: Define the one-step defect. Starting with the exact value e^{t_n} , the difference between the exact next value and one Euler step is

$$d_n = e^{t_n+h} - e^{t_n}(1 + h) = e^{t_n}(e^h - 1 - h) = \frac{1}{2}e^{t_n}h^2 + O(h^3).$$

On $0 \leq t_n \leq 1$, this is uniformly $O(h^2)$. Here defect means an error in solution value; dividing it by h would use a different convention.

Step 3: Derive the accumulated error. Because $Nh = 1$,

$$\ln Y_N = \frac{\ln(1 + h)}{h} = 1 - \frac{h}{2} + O(h^2).$$

Exponentiating gives $Y_N = e(1 - h/2 + O(h^2))$, so

$$e - Y_N = \frac{e}{2}h + O(h^2).$$

There are $N = 1/h$ steps over the fixed interval, and their defects propagate through later updates. An $O(h^2)$ one-step defect therefore need not yield an $O(h^2)$ endpoint error; here the accumulated error is first order.

Step 4: Compare the grids.

N	Y_N	$e - Y_N$
2	2.250000	0.468282
4	2.441406	0.276876
8	2.565785	0.152497

The errors approach a factor-of-two reduction under each doubling of N , because their leading term is proportional to h . Exact halving is not promised on coarse grids. The number of slope evaluations doubles, exposing the first-order accuracy versus work tradeoff.

Question 3

Consider a decaying quantity satisfying

$$y' = -4y, \quad y(0) = 1.$$

Apply explicit Euler with a constant step $h > 0$ for arbitrarily many steps.

Tasks

1. Derive the numerical solution Y_n and determine exactly which steps give $Y_n \rightarrow 0$ as $n \rightarrow \infty$.
2. Determine the stricter step condition that keeps every numerical value nonnegative and the sequence nonincreasing. Discuss $h = 1/4$ and $h = 1/2$ separately.
3. Compute enough values for $h = 0.4$ and $h = 0.6$ to show their contrasting behavior, and compare them graphically with the exact decay.
4. Explain why decreasing the physical solution's magnitude does not by itself ensure decay of an Euler approximation, and distinguish stability from faithful positivity.

Question 3 – Solution

Strategy. Analyze the discrete amplification factor rather than transferring the differential equation's stability directly to its numerical update.

Step 1: Solve the recurrence. The update is $Y_{n+1} = (1 - 4h)Y_n$, so $Y_n = (1 - 4h)^n$. It tends to zero exactly when

$$|1 - 4h| < 1, \quad 0 < h < 1/2.$$

Step 2: Impose positivity and monotonicity. All values are nonnegative and nonincreasing exactly when $0 \leq 1 - 4h < 1$, or

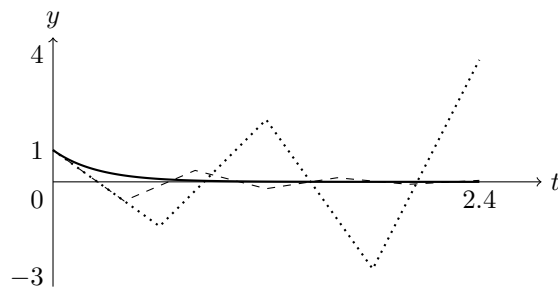
$$0 < h \leq 1/4.$$

At $h = 1/4$, the approximation becomes zero after one step, although e^{-4t} is strictly positive at every finite time. At $h = 1/2$, $Y_n = (-1)^n$ oscillates without decay. For $h > 1/2$, its magnitude grows without bound.

Step 3: Compare two oscillating sequences.

n	0	1	2	3	4
$h = 0.4$	1	-0.6	0.36	-0.216	0.1296
$h = 0.6$	1	-1.4	1.96	-2.744	3.8416

Each row is plotted at its own times $t_n = nh$, not at common node indices interpreted as time.



Solid: exact; dashed: $h = 0.4$; dotted: $h = 0.6$

Step 4: Interpret numerical stability. The exact solution $y = e^{-4t}$ decays smoothly. Euler replaces its per-step factor e^{-4h} by $1 - 4h$, which can be negative or have magnitude exceeding one. At $h = 0.4$ the method is asymptotically decaying but produces unphysical sign changes; at $h = 0.6$ it also loses decay. Stability controls long-run amplification, while positivity is an additional property with a stricter step requirement.

Question 4

A forced response obeys

$$y' = g(t) - y, \quad y(0) = 0, \quad g(t) = \begin{cases} 1, & 0 \leq t < 0.6, \\ 3, & t \geq 0.6. \end{cases}$$

The solution is continuous at the forcing switch; the differential equation is solved classically on each side. On any grid use $Y_{n+1} = Y_n + (t_{n+1} - t_n)[g(t_n) - Y_n]$.

Tasks

1. Compute Euler values on grid A: 0, 0.4, 0.8, 1.
2. Compute Euler values on grid B: 0, 0.4, 0.6, 1. Use the specified value of g at the switching node.
3. Derive the exact solution by matching the two phases and compare the two absolute endpoint errors.
4. Draw both Euler polygons and the exact solution. Explain why including the switch is important, yet does not guarantee the smaller endpoint error for these particular grids.

Question 4 – Solution

Strategy. Distinguish resolving an event from controlling the numerical error made within each phase.

Step 1: Compute grid A. The step lengths are 0.4, 0.4, 0.2. Left-node evaluation gives

$$Y(0.4) = 0.4, \quad Y(0.8) = 0.4 + 0.4(1 - 0.4) = 0.64, \quad \boxed{Y_A(1) = 0.64 + 0.2(3 - 0.64) = 1.112}.$$

The second step continues using $g = 1$ across an interval whose forcing actually changes partway through.

Step 2: Compute grid B. The step lengths are 0.4, 0.2, 0.4. Thus

$$Y(0.4) = 0.4, \quad Y(0.6) = 0.4 + 0.2(1 - 0.4) = 0.52, \quad \boxed{Y_B(1) = 0.52 + 0.4(3 - 0.52) = 1.512}.$$

At $t = 0.6$ the prescribed forcing is 3, so the final step uses the new phase.

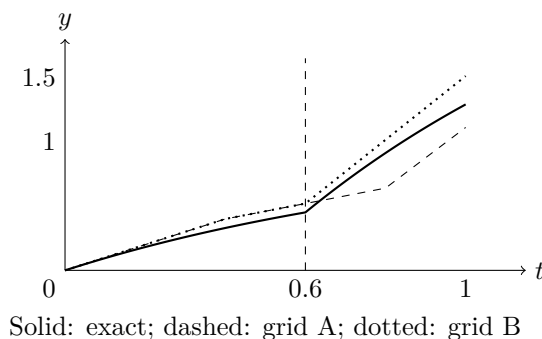
Step 3: Obtain the correct comparison value. The exact phase solutions are

$$y(t) = \begin{cases} 1 - e^{-t}, & 0 \leq t \leq 0.6, \\ 3 + (-2 - e^{-0.6})e^{-(t-0.6)}, & t \geq 0.6. \end{cases}$$

They agree at the switch and satisfy their respective equations. Consequently

$$y(1) = 3 - 2e^{-0.4} - e^{-1} \approx 1.291480,$$

so the absolute errors are $\boxed{0.179480 \text{ for A, } 0.220520 \text{ for B}}.$



Step 4: Explain the apparently better unaligned grid. Grid A undercounts the switched forcing, while grid B uses it correctly but takes a larger last step and overestimates the concave-down response. Here A happens to have the smaller endpoint error. Event alignment removes a known forcing mismatch, but does not order total errors on different coarse grids. Accurate computation also requires sufficiently small steps within the smooth phases.

Question 5

A normalized population satisfies

$$p' = 2p(1 - p), \quad 0 \leq p(0) \leq 1.$$

The exact solution preserves the physical interval $[0, 1]$. For explicit Euler with step $h > 0$, write $P_{n+1} = F_h(P_n)$. For the local discrete comparison, a small perturbation near $P = 1$ is multiplied to first order by $F'_h(1)$; call the linearized update contracting when $|F'_h(1)| < 1$.

Tasks

1. Derive F_h and find exactly which positive steps map all of $[0, 1]$ back into $[0, 1]$.
2. Prove the condition is necessary, not merely sufficient, by producing an overshooting initial value whenever it fails.
3. Compute the first two Euler updates for $h = 1$, $P_0 = 3/4$, and explain why they misrepresent the exact solution's qualitative behavior.
4. Find the step range for linearized contraction at 1. Compare it with interval preservation, and sketch the update maps for $h = 1/4$ and $h = 1$.

Question 5 – Solution

Strategy. Check the whole physical interval, not only the behavior of tiny perturbations near its upper endpoint.

Step 1: Factor the interval constraint. Euler gives $F_h(p) = p + 2hp(1-p)$. On $[0, 1]$ it is nonnegative. Also

$$1 - F_h(p) = (1-p)(1-2hp).$$

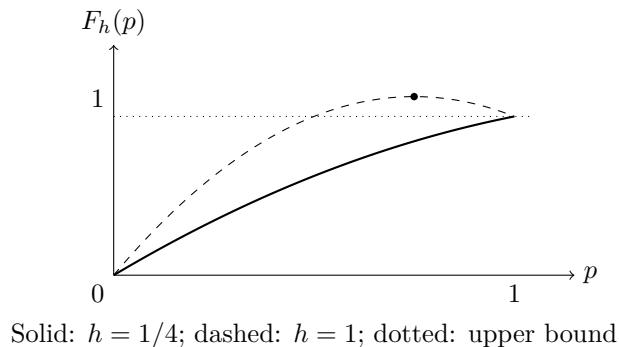
If $0 < h \leq 1/2$, both factors are nonnegative for every $p \in [0, 1]$, so $F_h(p) \leq 1$. Therefore $0 < h \leq 1/2$ preserves the interval at every step by induction.

Step 2: Prove necessity. If $h > 1/2$, choose p strictly between $1/(2h)$ and 1. Then $1-p > 0$ but $1-2hp < 0$, so $F_h(p) > 1$. Such a value exists for every $h > 1/2$, proving the stated condition is exact.

Step 3: Exhibit overshoot. For $h = 1$, $F_1(p) = 3p - 2p^2$, giving

$$P_1 = 9/8 = 1.125, \quad P_2 = 27/32 = 0.84375.$$

The numerical solution crosses above the capacity and then drops. The exact solution starting at $3/4$ increases toward 1 from below and cannot cross that equilibrium, by uniqueness.



Step 4: Distinguish two step restrictions. Since $F'_h(p) = 1 + 2h - 4hp$, $F'_h(1) = 1 - 2h$. Linearized contraction therefore holds for $0 < h < 1$, a larger range than interval preservation. At $h = 1$ the linearized magnitude equals one, so the strict contraction test makes no decision about nonlinear asymptotic behavior.

A locally contracting update can still move some valid states above the physical bound. The map criterion protects the entire interval; the derivative criterion only describes first-order behavior near one fixed point.

Question 6

Consider

$$y' = y^2, \quad y(0) = 1.$$

Two algorithms use a grid with $h = 1/4$. Algorithm A applies explicit Euler directly to y . Algorithm B first sets $u = 1/y$, applies explicit Euler to the transformed equation, and then recovers $y = 1/u$ wherever possible.

Tasks

1. Derive both update rules and compute their approximations at $t = 1/4$ and $t = 1/2$.
2. Find the exact IVP solution and determine which algorithm is exact at these nodes. Explain why this happens here.
3. Starting from a common positive value y with $hy < 1$, compare the two one-step updates algebraically and find their exact difference.
4. Explain what happens to algorithm B at $t = 1$ and why finite direct-Euler values at or beyond that time cannot establish continuation of the original IVP.

Question 6 – Solution

Strategy. A nonlinear change of dependent variable preserves the exact differential equation but generally does not commute with a finite Euler step.

Step 1: Derive and compute the algorithms. Direct Euler gives $Y_{n+1} = Y_n + hY_n^2$, $Y_0 = 1$. Since $u' = -y'/y^2 = -1$, transformed Euler gives $U_{n+1} = U_n - h$, $U_0 = 1$. Thus

t	A: direct Euler	B: $1/U_n$
$1/4$	$5/4$	$4/3$
$1/2$	$105/64 = 1.640625$	2

Step 2: Compare with the exact solution. Separation gives $y = 1/(1-t)$, whose maximal interval containing zero is $(-\infty, 1)$. Algorithm B is exact at every grid node before 1, because its transformed equation has the constant slope -1 : Euler integrates that affine solution exactly. Algorithm A underestimates the two displayed exact values.

Step 3: Compare a single step. From the same positive state, the updates are

$$A(y) = y + hy^2, \quad B(y) = \frac{1}{1/y - h} = \frac{y}{1 - hy}.$$

Their exact difference is

$$B(y) - A(y) = \frac{h^2 y^3}{1 - hy} > 0 \quad (hy < 1).$$

For fixed y , the discrepancy is of order h^2 as $h \rightarrow 0$, although both are Euler approximations in their chosen coordinates. The nonlinear reciprocal map changes the finite-step rule.

Step 4: Respect the original blow-up. At $t = 1$, algorithm B has $U_4 = 0$, so its inverse is undefined, matching the exact pole. The direct recurrence uses only addition and multiplication and gives a finite real value after any fixed finite number of exact-arithmetic steps. Those values are numerical iterates, not proof of a classical solution through the pole.

The original solution tends to $+\infty$ as $t \uparrow 1$, so it cannot extend continuously across that time. Beyond 1, an inverse value from negative U_n would likewise belong to a disconnected formula branch, not a continuation of this IVP.

Question 7

For $y' = 1 - y$, $y(0) = 0$, use explicit Euler to reach $t = 1$ with N equal steps, $h = 1/N$. Assume exact arithmetic and define the signed error $E_n = Y_n - y(t_n)$. You may use $0 \leq e^{-h} - 1 + h \leq h^2/2$ for $0 < h \leq 1$.

Tasks

1. Find the exact solution and a closed formula for the Euler sequence. Determine the sign of E_n .
2. Derive an error recurrence using the one-step defect evaluated from the exact solution.
3. Prove $0 \leq E_n \leq (h/2)[1 - (1 - h)^n] \leq h/2$. Use the simpler last bound to choose an integer N guaranteeing endpoint error at most 0.005.
4. For your chosen N , calculate the actual endpoint error from the closed formulas. Explain the difference between a guaranteed bound and an observed error, and state which arithmetic assumption the proof uses.

Question 7 – Solution

Strategy. Exploit the contracting error recurrence to turn a local defect bound into a rigorous global guarantee.

Step 1: Solve both continuous and discrete problems. The exact solution is $y(t) = 1 - e^{-t}$. Euler gives $Y_{n+1} = (1 - h)Y_n + h$, so

$$\boxed{Y_n = 1 - (1 - h)^n}, \quad E_n = e^{-nh} - (1 - h)^n \geq 0.$$

For $0 < h < 1$, the sign follows from $1 - h < e^{-h}$; at $h = 1$, it follows directly for $n \geq 1$. At $n = 0$, both solutions are zero.

Step 2: Isolate the local defect. Starting Euler at the exact value at t_n , its overshoot of the exact next value is

$$d_n = y(t_n) + h[1 - y(t_n)] - y(t_n + h) = e^{-t_n}(e^{-h} - 1 + h).$$

Therefore $0 \leq d_n \leq h^2/2$, and subtracting the exact update from the numerical one gives

$$\boxed{E_{n+1} = (1 - h)E_n + d_n, \quad E_0 = 0}.$$

Step 3: Sum the propagated defects. Since $0 \leq 1 - h < 1$, iteration gives

$$0 \leq E_n \leq \frac{h^2}{2} \sum_{j=0}^{n-1} (1 - h)^j = \frac{h}{2} [1 - (1 - h)^n] \leq \frac{h}{2}.$$

Thus $1/(2N) \leq 0.005$ is sufficient. The smallest integer supplied by this simpler bound is $\boxed{N = 100}$, with $h = 0.01$. This is not a claim that no smaller N could meet the tolerance using a sharper analysis.

Step 4: Evaluate the actual error. For that grid,

$$\boxed{E_{100} = e^{-1} - (0.99)^{100} \approx 0.00184710 < 0.005}.$$

The proven bound controls the error without assuming a particular observed cancellation or leading asymptotic term. Its conservatism explains the smaller actual error. Both the recurrence and this guarantee assumed exact arithmetic; step-by-step rounding would add another error term and require a separate bound.

Question 8

For $y' = -y$, $y(0) = 1$, let Y_h denote the Euler approximation at $t = 1$ with equal step h . Compare $h = 1/2$ and $h = 1/4$. Assume exact arithmetic. For the extrapolation argument, suppose $Y_h = y(1) + Ch + O(h^2)$ with a coefficient C independent of h as $h \rightarrow 0$.

Tasks

1. Compute $Y_{1/2}$, $Y_{1/4}$, and $R = 2Y_{1/4} - Y_{1/2}$.
2. Use the stated error expansion to explain why R cancels the leading first-order error. Relate the difference between the two Euler values to an estimate of the finer-grid error.
3. Compare the actual errors of the two Euler values and R with the exact endpoint. Is it a contradiction if R lies outside the interval between the Euler values?
4. Test whether grid agreement certifies accuracy using a different IVP, $z' = \sin^2(4\pi t)$, $z(0) = 0$. Compute both Euler endpoints on the same grids and the exact value $z(1)$, and explain the failure of the grid-difference indicator.

Question 8 – Solution

Strategy. Separate asymptotic error cancellation from a rigorous certificate: two grids can share the same sampling blind spot.

Step 1: Compute the three approximations. For decay, $Y_h = (1 - h)^{1/h}$. Hence

$$Y_{1/2} = 1/4, \quad Y_{1/4} = 81/256, \quad R = 49/128 = 0.3828125.$$

Step 2: Cancel the leading error term. The assumed expansions give

$$2Y_{h/2} - Y_h = y(1) + O(h^2), \quad Y_{h/2} - Y_h = -Ch/2 + O(h^2).$$

Thus $|Y_{h/2} - Y_h|$ estimates the leading magnitude of the finer-grid error, while the extrapolated value removes that term. This argument is asymptotic; it supplies no explicit bound for the omitted terms on these particular grids.

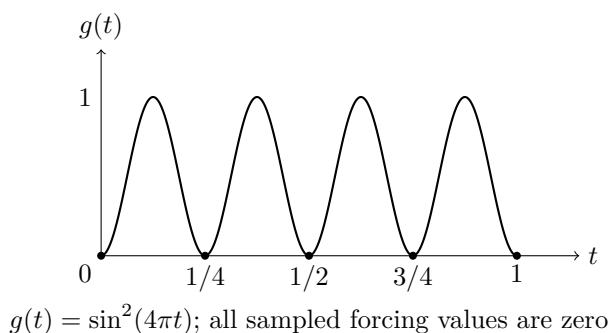
Step 3: Compare actual errors. The exact endpoint is e^{-1} . Absolute errors are

value	$Y_{1/2}$	$Y_{1/4}$	R
absolute error	0.117879	0.051473	0.014933

The finer-grid estimate is $17/256 = 0.06640625$, not its exact error. Both Euler values are below the truth, while R is above it. Extrapolation is not interpolation, so lying outside their interval is consistent with its construction and does not by itself invalidate it.

Step 4: Exhibit shared undersampling. At all left nodes of both specified grids, $\sin^2(4\pi t) = 0$. Both Euler approximations for $z(1)$ are therefore zero, and their difference is zero. But

$$z(1) = \int_0^1 \sin^2(4\pi t) dt = \frac{1}{2}.$$



The nonnegative forcing is missed between the sample nodes. Agreement of two grids does not certify accuracy; an explicit error bound or further analysis of the resolved time scales is needed.

Question 9

Consider $y' = -0.1y$, $y(0) = 1$, with Euler step $h = 1$. Let Z_n be the exact-arithmetic Euler sequence. A second implementation rounds each new value to the nearest tenth immediately after each step, calling the result Y_n . Use decimal rounding with ties away from zero; in particular, 0.45 rounds to 0.5. Assume the stated decimal rule is applied exactly, rather than relying on binary floating-point tie behavior.

Tasks

1. Derive the exact-arithmetic and rounded recurrences. Tabulate Y_n for $n = 0, 1, \dots, 8$ and compare its eventual behavior with Z_n and the exact solution.
2. Find every nonnegative multiple of 0.1 that is a fixed point of the rounded update.
3. Writing each rounding error as ρ_n , derive an error recurrence for $D_n = Y_n - Z_n$ and prove $|D_n| \leq 0.5(1 - 0.9^n)$.
4. Explain why a stable Euler amplification factor does not force a repeatedly rounded computation to converge to the true equilibrium. Distinguish rounding at every step from rounding only the reported final answer.

Question 9 – Solution

Strategy. Treat rounding as a new perturbation at each step; it can create fixed points absent from the exact-arithmetic recurrence.

Step 1: Compute both updates. Exact Euler gives $Z_{n+1} = 0.9Z_n$, so $Z_n = 0.9^n$. The rounded implementation uses $Y_{n+1} = \text{round}_{0.1}(0.9Y_n)$ and yields

n	0	1	2	3	4	5	6	7	8
Y_n	1.0	0.9	0.8	0.7	0.6	0.5	0.5	0.5	0.5

Once it reaches 0.5, its unrounded next value is 0.45, which rounds back to 0.5. Both Z_n and the exact solution $e^{-0.1t}$ tend to zero, whereas this implementation stays at 0.5.

Step 2: Identify all spurious fixed points. Write a nonnegative grid value as $Y = k/10$, with integer $k \geq 0$. In units of one tenth, the unrounded update is $0.9k$. For $k > 0$, rounding back to k occurs exactly when

$$k - \frac{1}{2} \leq 0.9k < k + \frac{1}{2},$$

which reduces to $k \leq 5$. Zero is also fixed. Thus the complete set is

$$\boxed{0, 0.1, 0.2, 0.3, 0.4, 0.5}.$$

Step 3: Bound accumulated rounding. Writing $Y_{n+1} = 0.9Y_n + \rho_n$ gives $|\rho_n| \leq 0.05$. Therefore

$$D_{n+1} = 0.9D_n + \rho_n, \quad D_0 = 0,$$

so

$$\boxed{|D_n| \leq 0.05 \sum_{j=0}^{n-1} 0.9^j = 0.5(1 - 0.9^n)}.$$

This bound approaches a nonzero limit, consistent with the observed plateau.

Step 4: Interpret the stable but biased computation. A factor 0.9 damps old perturbations, but fresh rounding occurs at every step and can balance the intended decrease. Stability does not remove continually injected errors. Rounding only the final reported answer would leave the preceding exact-arithmetic Euler trajectory unchanged and would introduce only one final reporting error, rather than altering subsequent slopes.

Question 10

Use explicit Euler to integrate $y' = -2y$ from $t = 1$ backward to $t = 0$. Begin with the benchmark terminal value $y(1) = e^{-2}$ and use time increments $\Delta t = -1/4$. This is the explicit Euler formula on a decreasing time grid, not the implicit method sometimes called backward Euler.

Tasks

1. Derive the update on the decreasing grid and compute the approximation at $t = 0$.
2. Find the benchmark exact solution, calculate the reconstruction error at 0, and sketch the numerical polygon with its direction of computation.
3. Replace the supplied terminal value by $e^{-2} + \delta$. Determine how this perturbation affects the numerical reconstruction and the exact reconstruction.
4. Now suppose e^{-2} is a reported terminal measurement with unknown error at most 0.001. Find the interval of possible exact initial values. Can reducing the Euler step alone guarantee reconstruction within 0.005 of the unknown true initial value? Explain.

Question 10 – Solution

Strategy. Track the sign of the time increment and distinguish discretization error from amplification of uncertain terminal data.

Step 1: March backward explicitly. Writing $t_j = 1 - j/4$, Euler gives

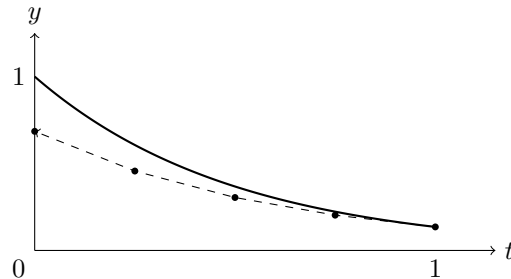
$$Y_{j+1} = Y_j - \frac{1}{4}(-2Y_j) = \frac{3}{2}Y_j.$$

After four steps,

$$Y(0) = \left(\frac{3}{2}\right)^4 e^{-2} = \frac{81}{16}e^{-2} \approx 0.685135.$$

Step 2: Compare with the exact benchmark. The exact solution is $y(t) = e^{-2t}$, giving $y(0) = 1$. Thus the numerical reconstruction error is

$$Y(0) - y(0) = \frac{81}{16}e^{-2} - 1 \approx -0.314865.$$



Solid: exact; dashed: Euler computed from $t = 1$ to $t = 0$

Each decreasing-time step amplifies the current value by 1.5. The exact amplification over a quarter-unit backward step is $e^{1/2} > 1.5$, explaining the numerical underestimate.

Step 3: Propagate a data perturbation. Linearity shows that replacing the terminal value by $e^{-2} + \delta$ changes the numerical reconstruction by

$$\left(\frac{81}{16}\right)\delta,$$

whereas exact backward evolution changes it by $e^2\delta$. With N backward steps of size magnitude $1/N$, the numerical amplification is $(1 + 2/N)^N$, which tends to e^2 , not to zero. Refinement improves the evolution calculation but does not eliminate sensitivity to data.

Step 4: Separate the uncertainty floor. The possible exact initial values form

$$[1 - 0.001e^2, 1 + 0.001e^2] \approx [0.992611, 1.007389].$$

Its half-width is about 0.007389, exceeding 0.005. No single reconstruction from the reported value can be within 0.005 of every possible true initial value in that interval. Reducing the step alone therefore cannot supply the requested guarantee; the terminal data would also need tighter uncertainty. Forward physical decay and backward data amplification describe different directions of evolution.