

Question 1

Consider the following equations. In (A)–(D), y is a function of x ; in (E), u is a function of the independent variables x and t .

(A) $x^2 y'' + \sin(x) y' - 3y = e^x$,

(B) $y'' + y y' = 0$,

(C) $(y')^2 + y = x$,

(D) $y'' + \sin(y) = 0$,

(E) $u_t = 4u_{xx} + x u$.

Tasks

1. Identify each equation as an ordinary or partial differential equation, and state its order.
2. Decide whether each equation is linear in its unknown function. For every nonlinear equation, identify the precise obstruction to linearity.
3. Among the linear equations, identify which are homogeneous after all unknown-function terms are moved to the left.
4. Explain why the factor x^2 in (A) and the factor x in (E) do not cause nonlinearity.

Question 1 – Solution

Strategy. Count the highest derivative order, then examine how the unknown function and its derivatives occur. A linear equation has coefficients depending only on independent variables.

Step 1: Classification.

Equation	Type	Order	Linearity
(A)	ODE	2	Linear
(B)	ODE	2	Nonlinear: product $y y'$
(C)	ODE	1	Nonlinear: square $(y')^2$
(D)	ODE	2	Nonlinear: $\sin(y)$
(E)	PDE	2	Linear

In (E), u_t has order one and u_{xx} has order two; the highest order is therefore two. In (C), squaring a first derivative does not make it a second derivative.

Step 2: Homogeneity. The two linear equations can be written as

$$x^2 y'' + \sin(x) y' - 3y = e^x, \quad u_t - 4u_{xx} - xu = 0.$$

Thus (A) is **nonhomogeneous**, and (E) is **homogeneous**. Here “homogeneous” is being used in the linear-equation sense.

Step 3: Coefficients versus unknowns. The factors x^2 , $\sin(x)$, and x are specified functions of independent variables, so they are allowed coefficients. By contrast, y multiplying y' in (B) is part of the unknown solution.

Check. Order and linearity are different properties: (C) is first order but nonlinear, whereas (A) is second order but linear. The vanishing of x^2 at 0 affects normalization of (A), not this linearity classification.

Question 2

For the initial value problem

$$y' + 2y = 6, \quad y(0) = 1,$$

three candidate functions are proposed:

$$f(x) = 3 - 2e^{-2x}, \quad g(x) = 3 - 2e^{2x}, \quad h(x) = 3.$$

Tasks

1. For each candidate, calculate its residual $R[v] = v' + 2v - 6$ and its value at $x = 0$.
2. Determine which candidates solve the differential equation and which solve the initial value problem. State an interval for each accepted solution.
3. Within the family $y = 3 + Ce^{-2x}$, determine the parameter selected by the initial condition.
4. Explain why matching one initial value, or satisfying an equation at a single point, does not verify a solution of an initial value problem.

Question 2 – Solution

Strategy. Verify two separate requirements: the differential equation must hold throughout an interval, and the initial value must hold at the specified point.

Step 1: Residuals. Direct differentiation gives

$$\begin{aligned}f' &= 4e^{-2x}, & R[f] &= 4e^{-2x} + 6 - 4e^{-2x} - 6 = 0, \\g' &= -4e^{2x}, & R[g] &= -4e^{2x} + 6 - 4e^{2x} - 6 = -8e^{2x}, \\h' &= 0, & R[h] &= 0 + 6 - 6 = 0.\end{aligned}$$

Also $f(0) = 1$, $g(0) = 1$, and $h(0) = 3$.

Step 2: Decisions. Since $e^{2x} > 0$, the residual for g never vanishes. Hence

f and h solve the differential equation on $\mathbb{R}$,
 f alone among these candidates solves the IVP on $\mathbb{R}$.

The function g passes the initial-value test but fails the equation; h does the reverse.

Step 3: Parameter selection. For $y = 3 + Ce^{-2x}$,

$$y' = -2Ce^{-2x}, \quad y' + 2y = 6.$$

Every real C gives an equation solution on $\mathbb{R}$. The initial condition requires $3 + C = 1$, so $C = -2$, recovering f .

Step 4: Logical distinction. An initial value is one pointwise constraint. A differential equation is an identity along the candidate function on an interval. For example, $v(x) = 3 + x^2$ satisfies $v' + 2v = 6$ at $x = 0$, but its residual $2x + 2x^2$ is not identically zero on any open interval. Neither pointwise test replaces interval verification.

Question 3

An implicit relation and an initial value are given by

$$x^2 + y^2 = 9, \quad y(0) = 3.$$

The proposed differential equation is

$$y' = -\frac{x}{y}.$$

Tasks

1. Identify the explicit branch selected by the initial value and verify the differential equation.
2. Find the largest open interval containing 0 on which that branch is a classical solution of the displayed equation.
3. Explain why the entire circle is not one solution function $y(x)$, and why adding the endpoints of the upper semicircle does not extend the solution interval.
4. If the initial value is changed to $y(0) = -3$, identify the corresponding branch and its interval. Sketch both branches in your solution.

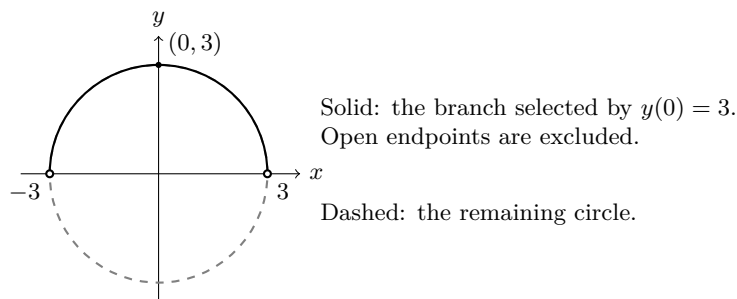
Question 3 – Solution

Strategy. An implicit curve can contain multiple function branches. Select the branch using the initial value, then check both differentiability and the equation's denominator.

Step 1: Select and verify. Solving for y gives $y = \pm\sqrt{9-x^2}$. Since $y(0) = 3$, choose

$$y(x) = \sqrt{9-x^2}, \quad -3 < x < 3.$$

On this interval, $y' = -x/\sqrt{9-x^2} = -x/y$, and $y(0) = 3$.



Step 2: Interval and endpoints. At $x = \pm 3$, the branch has $y = 0$, so $-x/y$ is undefined and the derivative is not finite. For $|x| > 3$, this branch is not real. Thus the largest open interval containing 0 is exactly $(-3, 3)$.

Step 3: Relation versus function. For each $-3 < x < 3$, the circle has two distinct y -values. The whole circle fails the single-valued requirement for a function $y(x)$. Even the undivided equation $yy' = -x$ cannot hold with a finite derivative at $(\pm 3, 0)$: it would require $0 = \mp 3$.

Step 4: Other initial value. For $y(0) = -3$,

$$y(x) = -\sqrt{9-x^2}, \quad -3 < x < 3.$$

Its derivative is $x/\sqrt{9-x^2} = -x/y$, verifying the lower branch separately.

Question 4

Consider

$$y' = y(1 - y).$$

A student divides by $y(1 - y)$ and claims that

$$\frac{y'}{y(1 - y)} = 1$$

is an equivalent equation for every solution. The following functions are available for testing:

$$y_0(x) = 0, \quad y_1(x) = 1, \quad p(x) = \frac{e^x}{1 + e^x}.$$

Tasks

1. Find all constant solutions of the original equation.
2. Verify p directly and state its domain and range.
3. Explain exactly when the division is valid and which listed solutions it excludes.
4. Match the initial values $y(0) = 0$, $y(0) = 1$, and $y(0) = \frac{1}{2}$ to the listed candidates. Explain why this matching alone is not a proof of uniqueness among all possible solutions.

Question 4 – Solution

Strategy. Check the original equation before transforming it. Division by an expression involving the unknown can discard legitimate solutions.

Step 1: Constant solutions. If $y(x) = k$, then $y' = 0$, so

$$0 = k(1 - k) \iff \boxed{k = 0 \text{ or } k = 1.}$$

Both constant functions solve the equation on $\mathbb{R}$; these are its equilibrium solutions.

Step 2: Nonconstant verification. The quotient rule gives

$$p'(x) = \frac{e^x(1 + e^x) - e^{2x}}{(1 + e^x)^2} = \frac{e^x}{(1 + e^x)^2}.$$

Also

$$p(1 - p) = \frac{e^x}{1 + e^x} \frac{1}{1 + e^x} = p'.$$

The denominator is positive for every real x . Because e^x ranges over $(0, \infty)$,

$$\boxed{\text{domain}(p) = \mathbb{R}, \quad \text{range}(p) = (0, 1).}$$

Step 3: Restricted equivalence. Division is valid on an interval only if $y(x) \neq 0$ and $y(x) \neq 1$ everywhere on that interval. Thus it is valid for p , but is undefined for both equilibrium solutions. The equations are equivalent under the nonvanishing restriction, not for every original solution.

Step 4: Initial data and scope. Direct substitution yields

$$\boxed{y_0(0) = 0, \quad y_1(0) = 1, \quad p(0) = \frac{1}{2}.}$$

Each is therefore a verified solution of the corresponding IVP. Testing three candidates does not exclude every other function: existence of a matching candidate and uniqueness of an IVP are logically different claims.

Question 5

Consider the initial value problem

$$y' = 2\sqrt{y}, \quad y(0) = 0,$$

where the square root is the nonnegative real root. For each real parameter $a \geq 0$, define on $\mathbb{R}$

$$y_a(x) = \begin{cases} 0, & x \leq a, \\ (x - a)^2, & x > a. \end{cases}$$

Tasks

1. Prove that y_a is continuously differentiable at its joining point $x = a$.
2. Verify the differential equation everywhere, including at $x = a$, and verify the initial value.
3. Show that different parameters give distinct solutions of the same IVP. Identify an additional solution that never leaves 0.
4. Sketch y_0 , y_1 , and y_2 in your solution and explain what this example shows about uniqueness. Does it mean that every first-order IVP is nonunique?

Question 5 – Solution

Strategy. Check the joining point directly; solving on the two sides alone is insufficient. Then compare solutions with the same initial value.

Step 1: Smooth joining. Both pieces tend to 0 as $x \rightarrow a$. At a , the difference quotient is 0 for $h < 0$ and $h^2/h = h$ for $h > 0$. Hence $y'_a(a) = 0$ and

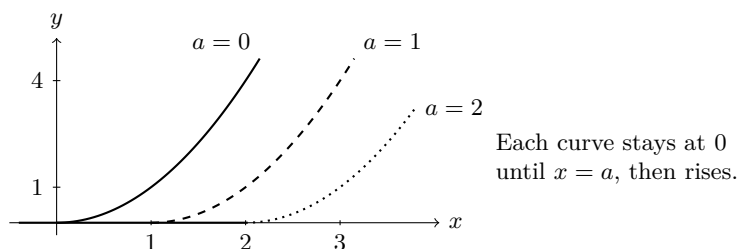
$$y'_a(x) = \begin{cases} 0, & x \leq a, \\ 2(x - a), & x > a. \end{cases}$$

This derivative is continuous at a , so y_a is C^1 on $\mathbb{R}$.

Step 2: Equation and initial value. For $x \leq a$, both y'_a and $2\sqrt{y_a}$ are 0. For $x > a$, $x - a > 0$, so

$$2\sqrt{(x - a)^2} = 2|x - a| = 2(x - a) = y'_a.$$

Since $a \geq 0$, $y_a(0) = 0$. Thus every y_a solves the IVP on $\mathbb{R}$.



Step 3: Distinctness. If $0 \leq a < b$, choose $a < x < b$. Then $y_a(x) > 0$ but $y_b(x) = 0$. Also $y \equiv 0$ is a solution distinct from every finite-parameter y_a .

Step 4: Meaning. This IVP has infinitely many solutions: the initial value does not determine when a solution leaves 0. It disproves automatic uniqueness for all first-order IVPs; it does not assert nonuniqueness for every IVP. No general uniqueness theorem is needed for this counterexample.

Question 6

A one-parameter family of curves is

$$y(x) = (x - C)^2, \quad C \in \mathbb{R}.$$

Tasks

1. Differentiate the family and eliminate C to obtain a first-order differential equation involving only y and y' .
2. Verify that every family member satisfies the resulting equation on $\mathbb{R}$.
3. Find all family members satisfying $y(0) = 1$, and compare their slopes at 0.
4. Test $y \equiv 0$ in your equation. Is it a member of the displayed family on a nonempty open interval? Explain why finding one family of solutions need not describe every solution.

Question 6 – Solution

Strategy. Use the derivative to remove the constant, then investigate what information was lost and whether new solutions are admitted.

Step 1: Eliminate the parameter. Differentiating yields $y' = 2(x - C)$. Squaring and using $y = (x - C)^2$ gives

$$(y')^2 = 4y.$$

This is a first-order nonlinear ODE. The exponent on y' changes linearity, not derivative order.

Step 2: Direct verification. For any real C ,

$$(y')^2 = [2(x - C)]^2 = 4(x - C)^2 = 4y.$$

The family members are polynomials and therefore satisfy the equation on all of $\mathbb{R}$, including $x = C$.

Step 3: Apply the initial value. At $x = 0$ we require $C^2 = 1$, so $C = \pm 1$. The two selected family members are

$$y = (x - 1)^2 \quad \text{and} \quad y = (x + 1)^2.$$

Their slopes at 0 are -2 and 2 , respectively. The equation itself permits either sign because it specifies $(y')^2$, not y' .

Step 4: A solution outside the family. The function $y \equiv 0$ satisfies $(y')^2 = 4y$ identically. If $(x - C)^2$ were identically 0 on a nonempty open interval, every x in that interval would have to equal one fixed number C , which is impossible. Thus

$$y \equiv 0 \text{ is a solution not contained in the displayed family.}$$

Elimination proves that family members solve the equation; it does not prove the converse. In particular, a parameterized family cannot be called a complete description merely because substitution succeeds.

Question 7

You are given that every real solution of $y'' + y = 0$ on $\mathbb{R}$ has the form

$$y(x) = A \cos x + B \sin x.$$

Consider three sets of conditions:

$$(I) \quad y(0) = 2, \quad y'(0) = -1,$$

$$(II) \quad y(0) = 2, \quad y(\pi) = -2,$$

$$(III) \quad y(0) = 2, \quad y(\pi) = 0.$$

Tasks

1. Verify the given family by differentiation.
2. Identify each problem as an initial value problem or a boundary value problem.
3. Determine whether each has exactly one solution, infinitely many solutions, or no solution. Give the corresponding functions when they exist.
4. Explain why the statement “two conditions determine a unique solution of a second-order equation” needs additional qualifications.

Question 7 – Solution

Strategy. Translate each condition into an equation for A and B . Conditions may be independent, redundant, or inconsistent.

Step 1: Verify the family. We have

$$y' = -A \sin x + B \cos x, \quad y'' = -A \cos x - B \sin x = -y.$$

Thus every family member solves $y'' + y = 0$ on $\mathbb{R}$.

Step 2: Initial conditions. In (I), both values are prescribed at $x = 0$, so this is an IVP. Since $y(0) = A$ and $y'(0) = B$,

$$A = 2, \quad B = -1, \quad \boxed{y = 2 \cos x - \sin x.}$$

The supplied completeness of the family makes this the unique solution.

Step 3: Boundary conditions. In (II) and (III), values are prescribed at two different points, so these are BVPs. In both, $y(0) = 2$ gives $A = 2$. But

$$y(\pi) = A \cos \pi + B \sin \pi = -A = -2,$$

independently of B . Consequently,

$$\boxed{\text{(II): } y = 2 \cos x + B \sin x, \quad B \in \mathbb{R}}$$

has infinitely many solutions. In (III), the demand $y(\pi) = 0$ contradicts $y(\pi) = -2$, so $\boxed{\text{(III) has no solution.}}$

Step 4: Why counting fails. Two written conditions need not provide two independent, consistent constraints. In (II) the second repeats information already forced by the first; in (III) it conflicts with that information. Their number alone cannot establish uniqueness or even existence.

Question 8

For a real parameter a , define

$$z_a(x) = \begin{cases} x^2, & x \leq 0, \\ ax^2, & x > 0. \end{cases}$$

Compare the equations

$$(A) \quad xy' = 2y, \quad (B) \quad y'' = 2.$$

A classical solution must have all derivatives appearing in its equation at every point of its open interval of definition.

Tasks

1. Determine for which a the function z_a is C^1 on $\mathbb{R}$, and for which a its second derivative exists at 0.
2. Determine for which a it solves (A) on $\mathbb{R}$, checking $x = 0$ explicitly.
3. Determine for which a it solves (B) on $\mathbb{R}$.
4. Explain why replacing (A) by $y' = 2y/x$ can change the permitted solution intervals, even though the two equations agree whenever $x \neq 0$.

Question 8 – Solution

Strategy. Compute derivatives at the joining point using limits, then test each equation in its original form.

Step 1: First derivative. The pieces meet at $z_a(0) = 0$. The difference quotient at 0 is h for $h < 0$ and ah for $h > 0$, so $z'_a(0) = 0$ for every real a . Therefore

$$z'_a(x) = \begin{cases} 2x, & x < 0, \\ 0, & x = 0, \\ 2ax, & x > 0, \end{cases}$$

which is continuous at 0. Hence $z_a \in C^1(\mathbb{R})$ for every $a \in \mathbb{R}$.

Step 2: Second derivative at the join. The left-hand limit of $[z'_a(h) - z'_a(0)]/h$ is 2, and the right-hand limit is $2a$. Thus

$$z''_a(0) \text{ exists exactly when } a = 1.$$

Step 3: Check equation (A). For $x < 0$, $xz'_a = x(2x) = 2z_a$. For $x > 0$, $xz'_a = x(2ax) = 2z_a$. At $x = 0$, both sides are 0. Consequently all real a give solutions of (A) on $\mathbb{R}$.

Step 4: Check equation (B). On $x > 0$, $z''_a = 2a$, so (B) requires $a = 1$. For that value $z_a = x^2$ globally and $z''_a = 2$, including at 0. Thus $a = 1$ is necessary and sufficient for (B).

Step 5: Domain of the equation. The quotient $2y/x$ is undefined at $x = 0$, even when $y(0) = 0$. The normalized equation therefore cannot have a solution interval containing 0. It agrees with (A) on intervals contained in $(-\infty, 0)$ or $(0, \infty)$, but division removes the original equation's valid point $x = 0$.

Question 9

For a function $u(x, t)$, consider the partial differential equation and initial profile

$$u_t = 4u_{xx}, \quad u(x, 0) = 3 \cos(2x), \quad x \in \mathbb{R}.$$

Search only within the candidate family

$$u(x, t) = Ae^{-\lambda t} \cos(kx), \quad A \neq 0, \quad k > 0,$$

where A , λ , and k are real constants.

Tasks

1. State the independent and dependent variables, the order of the PDE, and whether it is linear and homogeneous.
2. Calculate u_t and u_{xx} and determine exactly which relation among the parameters makes the family satisfy the PDE everywhere.
3. Use the initial profile to determine A , k , and λ , and verify the resulting function.
4. Explain why a condition specifying $u(x, 0)$ supplies more information than one scalar condition such as $u(0, 0) = 3$. Distinguish uniqueness within the proposed family from uniqueness among all PDE solutions.

Question 9 – Solution

Strategy. Treat the other independent variable as constant in each partial derivative. Verify the PDE before imposing the entire initial profile.

Step 1: Definitions. The independent variables are x, t and the dependent variable is u . Written as $u_t - 4u_{xx} = 0$, the equation is second order, linear, and homogeneous.

Step 2: Parameter relation. Direct differentiation gives

$$u_t = -\lambda A e^{-\lambda t} \cos(kx), \quad u_{xx} = -k^2 A e^{-\lambda t} \cos(kx).$$

Thus the residual is

$$u_t - 4u_{xx} = (4k^2 - \lambda) A e^{-\lambda t} \cos(kx).$$

At $x = 0$ the cosine equals 1, while $A e^{-\lambda t} \neq 0$. The residual vanishes everywhere if and only if $\lambda = 4k^2$; no division by a possibly zero cosine is needed.

Step 3: Initial profile. At $t = 0$, the condition is $A \cos(kx) = 3 \cos(2x)$ for every real x . Evaluation at $x = 0$ gives $A = 3$. Taking two x -derivatives of this identity at 0 gives $-Ak^2 = -12$, so $k^2 = 4$. Since $k > 0$, $k = 2$, and $\lambda = 16$. Therefore

$$u(x, t) = 3e^{-16t} \cos(2x).$$

For this function $u_t = -48e^{-16t} \cos(2x) = 4u_{xx}$, and the initial profile is recovered at $t = 0$. The function is smooth for all real x, t ; in particular it works for forward time $t \geq 0$.

Step 4: Amount of data. The profile prescribes a value for every x along $t = 0$. The single condition $u(0, 0) = 3$ fixes only A , leaving any $k > 0$ with $\lambda = 4k^2$ possible. The full profile selects one candidate in the stated family; no uniqueness assertion about all possible PDE solutions has been proved.

Question 10

Let $a \in \mathbb{R}$ be a fixed parameter in the equation

$$(a - 1)y'' + (a^2 - 1)(y')^2 + (a + 1)y' = x.$$

The proposed initial data are

$$y(0) = 0, \quad y'(0) = 0.$$

Tasks

1. Classify the equation's order and linearity for every real value of a . Simplify the exceptional cases before classifying them.
2. For each parameter value that makes the equation linear, find all solutions satisfying the proposed data using elementary integration.
3. For $a = 1$, decide whether $y'(0) = 0$ supplies independent information beyond the equation. What happens if this condition is replaced by $y'(0) = 1$?
4. Explain why the mere appearance of the symbol y'' in a parameterized formula does not guarantee that every member is a second-order equation.

Question 10 – Solution

Strategy. Determine when the leading coefficient or the nonlinear coefficient vanishes. Classify only after substituting those exceptional parameter values.

Step 1: Exhaustive classification. Factor $a^2 - 1 = (a - 1)(a + 1)$.

a	Simplified equation	Classification
1	$2y' = x$	First order, linear
-1	$-2y'' = x$	Second order, linear
$a \neq \pm 1$	Original equation	Second order, nonlinear

For $a \neq \pm 1$, both the y'' coefficient and the $(y')^2$ coefficient are nonzero. Both linear exceptional equations are nonhomogeneous because their right-hand sides are not identically zero.

Step 2: Solve when $a = 1$. From $y' = x/2$,

$$y = \frac{x^2}{4} + C, \quad y(0) = 0 \implies C = 0.$$

Thus $\boxed{y = x^2/4 \text{ on } \mathbb{R}}$. Its derivative at 0 is 0, as required, and $2y' = x$ verifies the equation directly.

Step 3: Solve when $a = -1$. Two integrations of $y'' = -x/2$ give

$$y' = -\frac{x^2}{4} + C_1, \quad y = -\frac{x^3}{12} + C_1x + C_2.$$

The data yield $C_1 = C_2 = 0$, so $\boxed{y = -x^3/12 \text{ on } \mathbb{R}}$. Indeed $-2y'' = x$ and both initial values are 0.

Step 4: Redundant or incompatible data. For $a = 1$, the equation at $x = 0$ already forces $2y'(0) = 0$. Thus the stated slope condition is redundant; replacing it with $y'(0) = 1$ makes the problem inconsistent. The coefficient $a - 1$ vanishes at $a = 1$, removing y'' entirely. Order is determined by derivatives with nonzero coefficients in the actual equation, not by unsimplified notation.

Question 1

Consider the direction field for

$$y' = x - y.$$

An *isocline of slope k* is the set of points where the equation assigns slope k . A direction field consists of short line segments with the prescribed slopes; the segments are not themselves whole solution curves.

Tasks

1. Calculate the slopes at $(-1, 1)$, $(0, 1)$, $(1, 1)$, and $(1, -1)$, and describe the orientation of each segment.
2. Find the isoclines of slopes -1 , 0 , and 1 . Determine the regions of positive and negative slope.
3. Draw a direction field on $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ in your solution. Mark the three isoclines and distinguish them from solution curves.
4. A solution passes through $(0, 1)$. Find its tangent line there and decide whether it initially rises or falls as x increases. Does the tangent line itself solve the equation on an interval?

Question 1 – Solution

Strategy. Evaluate $x - y$ at points, then hold its value fixed to find isoclines. Test a proposed curve by comparing its derivative with the assigned slope.

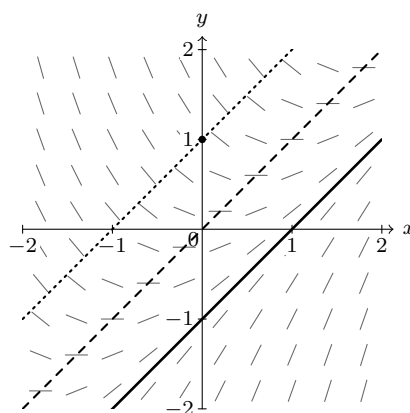
Step 1: Slopes and regions. The four slopes, in the listed order, are

$$\boxed{-2, \quad -1, \quad 0, \quad 2.}$$

Thus the first two segments fall to the right, the third is horizontal, and the fourth rises. The isocline equation is $x - y = k$, or $y = x - k$. Hence

$$k = -1 : y = x + 1, \quad k = 0 : y = x, \quad k = 1 : y = x - 1.$$

Below $y = x$ the slopes are positive; above it they are negative.



Solid: $y = x - 1$; dashed: $y = x$; dotted: $y = x + 1$. Dot: $(0, 1)$.

Step 2: Isoclines are not automatically solutions. Each displayed isocline has geometric slope 1. On $y = x + 1$ and $y = x$, the field instead prescribes -1 and 0 , so these lines are not solutions. On $y = x - 1$, the prescribed slope is 1 , so that particular isocline also happens to be a solution.

Step 3: Tangent versus solution. At $(0, 1)$ the slope is -1 , giving $\boxed{y = 1 - x}$ as the tangent line. The solution initially falls. Along this line, however, $x - y = 2x - 1$, which equals the line's derivative -1 only at $x = 0$. Thus the tangent line does not solve the equation on any open interval.

Question 2

Consider the autonomous equation

$$y' = y(2 - y).$$

You may use the following fact: for this smooth right-hand side, two solution curves cannot intersect at a common point unless they coincide on their common interval. Assume the solution with $y(0) = 1$ exists for all $x \geq 0$.

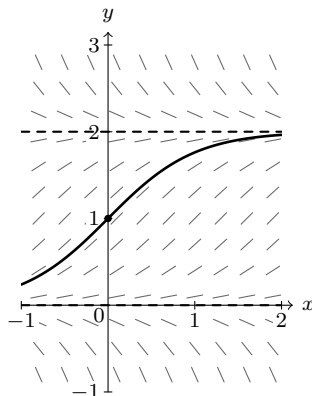
Tasks

1. Explain why the direction field repeats horizontally, and find all horizontal solution lines.
2. Determine the slope sign in each region separated by those lines. Draw the field for $-1 \leq x \leq 2$ and $-1 \leq y \leq 3$ in your solution.
3. Prove that the solution with $y(0) = 1$ remains between 0 and 2 and is increasing for $x \geq 0$.
4. Determine its limit as $x \rightarrow \infty$. Justify why a limiting value strictly below 2 would contradict the differential equation; do not solve the equation explicitly.

Question 2 – Solution

Strategy. Read the signs between horizontal zero-slope lines, then use nonintersection and a bounded-monotone argument.

Step 1: Field structure. The slope depends only on y , so it is the same along every horizontal row. Constant solutions require $y(2 - y) = 0$, giving $y = 0$ and $y = 2$. The sign is negative for $y < 0$, positive for $0 < y < 2$, and negative for $y > 2$.



Solid: solution through $(0, 1)$. Dashed: the equilibrium lines $y = 0, 2$.

Step 2: Confinement and monotonicity. Starting at 1, a continuous solution could leave $(0, 2)$ only by meeting one of the equilibrium solutions. The stated nonintersection fact excludes this. Therefore

$$0 < y(x) < 2, \quad y'(x) > 0 \quad (x \geq 0).$$

Since $y(0) = 1$, the function is bounded below by 1 on this half-line.

Step 3: Limit without an explicit formula. An increasing function bounded above by 2 has a limit L with $1 \leq L \leq 2$. If $L < 2$, continuity of $g(y) = y(2 - y)$ gives $g(L) > 0$ and, eventually, $y'(x) \geq g(L)/2 =: c > 0$. Integrating from a sufficiently large X yields

$$y(x) \geq y(X) + c(x - X),$$

contradicting the upper bound 2. Thus $\lim_{x \rightarrow \infty} y(x) = 2$. The limit is approached from below, without touching the equilibrium at any finite x .

Question 3

For the nonautonomous equation

$$y' = x + y,$$

a student claims that the zero-slope isocline is a solution because “every field segment on it is horizontal.” A candidate through the origin is supplied:

$$v(x) = e^x - x - 1.$$

Tasks

1. Find the zero-slope isocline and the regions where solutions rise or fall. Test the student’s claim.
2. Verify that v solves the equation and satisfies $v(0) = 0$.
3. Determine whether the horizontal tangent at the origin is a maximum, a minimum, or neither. Explain how v crosses the zero-slope isocline.
4. Draw the field, the isocline, and the verified solution near the origin in your solution. Explain why crossing an isocline does not contradict uniqueness of solution curves.

Question 3 – Solution

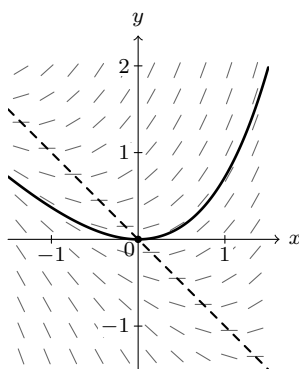
Strategy. Distinguish a line made of zero-slope locations from a curve whose own tangent agrees with the field everywhere.

Step 1: Zero slopes. The zero-slope isocline is $y = -x$. Slopes are positive above it and negative below it. The line itself has derivative -1 , while the equation assigns 0 along it, so it is not a solution.

Step 2: Verify the candidate. We have

$$v' = e^x - 1 = x + (e^x - x - 1) = x + v, \quad v(0) = 0.$$

Thus v is a solution on all of $\mathbb{R}$.



Solid: $v = e^x - x - 1$. Dashed: the zero-slope isocline $y = -x$.

Step 3: Classify the contact. Since $e^x - 1 < 0$ for $x < 0$ and $e^x - 1 > 0$ for $x > 0$, v decreases before 0 and increases after 0 . Therefore $(0, 0)$ is a strict minimum. Also $v'' = e^x > 0$ everywhere.

Step 4: Crossing the reference line. The signed vertical difference from $y = -x$ is

$$v(x) - (-x) = e^x - 1.$$

It changes from negative to positive at 0 , so the solution crosses the isocline from below to above. Nonintersection of distinct solutions is irrelevant: the isocline is not a solution curve.

Question 4

An unknown direction field is known to come from an affine law

$$y' = ax + by + c,$$

with fixed real constants a, b, c . Three exact slope measurements are

(x, y)	$(-1, 0)$	$(0, 1)$	$(1, 1)$
slope	-1	0	2

A fourth report claims that the slope at $(2, 0)$ is 4.

Tasks

1. Recover a, b, c and show that the first three measurements determine them uniquely.
2. Decide whether the fourth report is consistent with the model.
3. Find the zero-slope isocline. Then determine every straight-line solution $y = mx + d$ of the recovered equation.
4. Draw the recovered field, its zero-slope isocline, and its straight-line solution in your solution. Explain why the affine-model assumption matters when inferring a whole field from three points.

Question 4 – Solution

Strategy. Recover the coefficients from independent data, then compare the geometric slope of a line with the field along it.

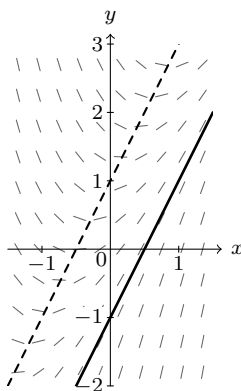
Step 1: Recover the law. The measurements give

$$-a + c = -1, \quad b + c = 0, \quad a + b + c = 2.$$

Subtracting the second equation from the third gives $a = 2$. The first then gives $c = 1$, and the second gives $b = -1$. These forced values prove uniqueness within the affine class:

$$y' = 2x - y + 1.$$

At $(2, 0)$ the predicted slope is 5, so the reported value 4 is inconsistent.



Solid: solution $y = 2x - 1$. Dashed: zero-slope isocline $y = 2x + 1$.

Step 2: Reference line and solution line. Zero slopes occur on $y = 2x + 1$, which is not a solution: its derivative is 2, not 0. Substituting $y = mx + d$ into the equation requires

$$m = (2 - m)x + 1 - d$$

for every x in an interval. Hence $m = 2$ and $d = -1$. The only straight-line solution is $y = 2x - 1$, verified by $2x - (2x - 1) + 1 = 2$.

Step 3: Limits of inference. Three data points determine the three affine coefficients here. Without the affine assumption, many nonlinear functions could agree at those points while assigning different slopes elsewhere. Finite measurements alone do not determine an unrestricted field.

Question 5

Consider the equation

$$y' = \frac{x}{y}, \quad y \neq 0,$$

and a solution through $(0, 1)$.

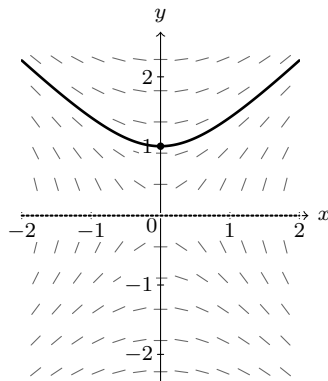
Tasks

1. Determine the slope sign in each open quadrant. Find every point in the domain with a horizontal field segment.
2. Explain why no segment should be placed at $(1, 0)$, even though nearby segments become steep.
3. Along a solution, differentiate $y^2 - x^2$. Use the initial point to identify an explicit solution branch and verify it directly. State its largest open interval.
4. Draw the field and the selected branch in your solution, marking the excluded line $y = 0$. Explain why that excluded line does not prevent this particular solution from existing for every real x .

Question 5 – Solution

Strategy. Respect the equation's domain before drawing. Use a constant quantity along a solution to identify the curve through the given point.

Step 1: Signs and missing points. Slopes are positive in Quadrants I and III, and negative in II and IV. Horizontal segments occur exactly where $x = 0$ and $y \neq 0$. The expression x/y is undefined on all of $y = 0$; a vertical mark at $(1, 0)$ would falsely assign a direction to a point outside the domain.



Solid: $y = \sqrt{1 + x^2}$. Dotted horizontal line: excluded $y = 0$.

Step 2: Identify and verify the branch. Wherever a solution is defined,

$$\frac{d}{dx}(y^2 - x^2) = 2yy' - 2x = 2y\frac{x}{y} - 2x = 0.$$

Thus $y^2 - x^2$ is constant on its interval. At $(0, 1)$ it equals 1. Continuity and the positive initial value select

$$y = \sqrt{1 + x^2}, \quad -\infty < x < \infty.$$

Indeed $y' = x/\sqrt{1 + x^2} = x/y$ and $y(0) = 1$.

Step 3: Interval and geometry. This branch is smooth and satisfies $y \geq 1$ for every real x , so it never reaches the excluded axis. The entire real line is therefore its largest open interval. It decreases for $x < 0$ and increases for $x > 0$, with a minimum at $(0, 1)$, matching the field.

Question 6

Compare two equations on the same coordinate scale:

$$(A) \quad y' = y, \quad (B) \quad y' = 3y.$$

Both are assigned the initial value $y(0) = 1$. Candidate solutions are supplied:

$$u(x) = e^x, \quad v(x) = e^{3x}.$$

Tasks

1. Determine the zero-slope locations and slope signs for both fields. Are these facts enough to make the two fields identical?
2. At height $y = 1$, compute each slope and its angle with the positive x -axis. If the two fields use equal-length segments, explain how steepness should be represented.
3. Verify the two candidates and calculate when each first reaches $y = 2$ for $x > 0$.
4. Draw the two fields and their respective verified curves on identical axes in your solution. State precisely which geometric information is lost by recording only slope signs.

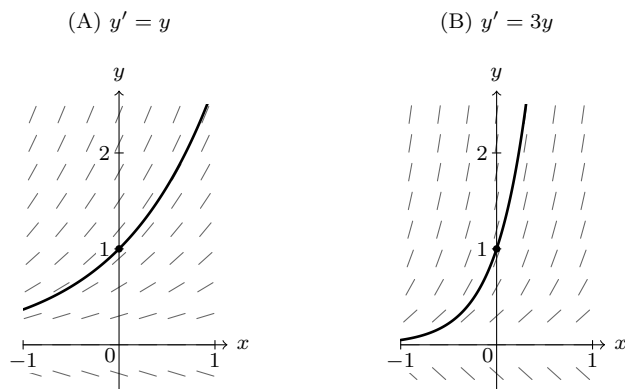
Question 6 – Solution

Strategy. Separate slope sign from slope magnitude. Equal-length field segments encode magnitude through their angle, not their length.

Step 1: Shared and different information. Both fields have horizontal segments on $y = 0$, positive slopes above it, and negative slopes below it. But at $y = 1$ their slopes are 1 and 3, so the fields are different. The corresponding angles are

$$\theta_A = \arctan(1) = 45^\circ, \quad \theta_B = \arctan(3) \approx 71.6^\circ.$$

A segment of total length ℓ and slope m can use the displacement $\ell(1, m)/\sqrt{1 + m^2}$. This preserves its slope while controlling its length.



Solid curves: e^x on the left and e^{3x} on the right; identical axis scales.

Step 2: Verify and compare growth. Directly,

$$u' = e^x = u, \quad v' = 3e^{3x} = 3v, \quad u(0) = v(0) = 1.$$

Both increase strictly. Solving $e^x = 2$ and $e^{3x} = 2$ gives their first hitting times

$$x_A = \ln 2, \quad x_B = \frac{\ln 2}{3}.$$

Thus (B) reaches the same height in one-third the time.

Step 3: What signs omit. Signs indicate rising, falling, or horizontal tangents. They do not specify tangent angles, rates of change, or travel times between heights. Identical sign regions and the same equilibrium line do not determine the same solution curves.

Question 7

For the equation

$$y' = x^2 - y,$$

consider the supplied candidate through the origin

$$w(x) = x^2 - 2x + 2 - 2e^{-x}.$$

You may use the calculus inequality $e^t > 1 + t$ for $t \neq 0$.

Tasks

1. Find the zero-slope isocline and describe the slope signs above and below it.
2. Verify that w solves the equation and that $w(0) = w'(0) = 0$.
3. Decide whether w has a local extremum at 0. Determine its concavity on either side of 0 and classify the horizontal tangent there.
4. Draw the field, the isocline, and w near the origin in your solution. Explain why observing a horizontal field segment is insufficient to conclude that a solution has a maximum or minimum.

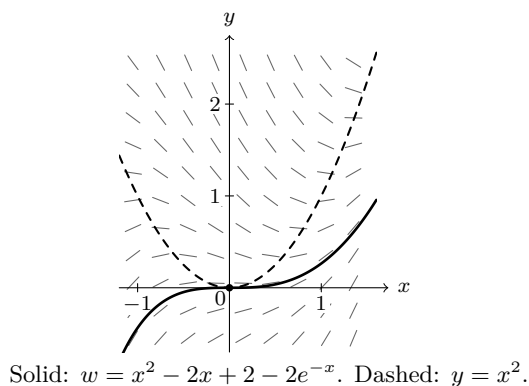
Question 7 – Solution

Strategy. A zero first derivative identifies a stationary point, not its type. Inspect nearby derivative signs and concavity.

Step 1: Isocline and verification. Zero slopes occur on $y = x^2$. Slopes are positive below this parabola and negative above it. For the candidate,

$$w' = 2x - 2 + 2e^{-x} = x^2 - w, \quad w(0) = w'(0) = 0.$$

It is therefore a solution on $\mathbb{R}$ with a horizontal tangent at the origin.



Step 2: No local extremum. For $x \neq 0$, the supplied inequality gives $e^{-x} > 1 - x$, so

$$w' = 2(x - 1 + e^{-x}) > 0.$$

The derivative is positive on both sides of 0. Thus w increases through the origin and has neither a local maximum nor a local minimum there. Moreover, $x^2 - w = w' > 0$ for $x \neq 0$, so the curve lies below its zero-slope isocline on both sides.

Step 3: Concavity. Since $w'' = 2 - 2e^{-x}$, the curve is concave down for $x < 0$ and concave up for $x > 0$. Hence

$$(0, 0) \text{ is an inflection point with a horizontal tangent.}$$

The field gives the tangent slope at a point; the surrounding behavior is needed to classify that point.

Question 8

Consider

$$y' = xy.$$

For any solution y on an interval I , define the reflected functions

$$g(x) = y(-x) \quad (x \in -I), \quad h(x) = -y(x) \quad (x \in I),$$

where $-I = \{-t : t \in I\}$. Also consider the candidate $p(x) = e^{x^2/2}$.

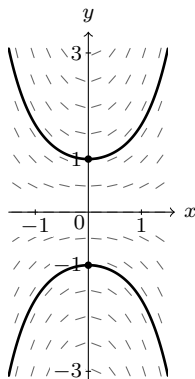
Tasks

1. Determine the zero-slope locations and the slope signs in the four quadrants.
2. Verify by differentiation that both reflected functions satisfy the same equation on their stated intervals. Explain why reflecting across the vertical axis requires a sign change in the derivative.
3. Verify p , find its minimum, and describe the corresponding extremum of $-p$.
4. Draw the field with p and $-p$ in your solution. Explain how the graph exhibits both reflection symmetries without treating a direction segment as an arrow giving a unique direction of travel.

Question 8 – Solution

Strategy. Check symmetry by transforming the differential equation itself, using the chain rule for reflection in the independent variable.

Step 1: Slopes. Since $xy = 0$ on both coordinate axes, all field segments there are horizontal. Slopes are positive in Quadrants I and III and negative in II and IV. The horizontal axis is an equilibrium solution; the vertical axis is not a function $y(x)$.



Solid: $p = e^{x^2/2}$ and $-p$. The field and curves respect both reflections.

Step 2: Transform the derivatives. For $x \in -I$, the chain rule gives

$$g'(x) = -y'(-x) = -[(-x)y(-x)] = xg(x).$$

The minus sign comes from differentiating $-x$. For $x \in I$,

$$h'(x) = -y'(x) = -xy(x) = xh(x).$$

Thus each reflection takes solutions to solutions on the stated reflected or original interval.

Step 3: Verify the displayed curves. We have $p' = xe^{x^2/2} = xp$ and $p(0) = 1$. Because $x^2/2 \geq 0$,

$\min_{\mathbb{R}} p = 1 \text{ at } x = 0,$	$\max_{\mathbb{R}}(-p) = -1 \text{ at } x = 0.$
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The curve p is even, and $-p$ is its reflection across the horizontal axis. Field segments indicate tangent lines. When following a graph in the direction of increasing x , the derivative sign specifies whether y rises or falls; the drawn segments themselves need no arrowheads.

Question 9

A plotting program samples slopes only at the five columns

$$x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$$

and at any chosen collection of y -values. Compare the two smooth equations

$$(A) \quad y' = 0, \quad (B) \quad y' = \sin(2\pi x).$$

Both have initial value $y(0) = 0$.

Tasks

1. Prove that the sampled direction fields are identical at every plotted point.
2. Add the column $x = \frac{1}{4}$. Calculate the slopes assigned by each law there and explain why it reveals the difference.
3. Verify the candidates $u(x) = 0$ and $v(x) = [1 - \cos(2\pi x)]/(2\pi)$ for the respective IVPs. Compare their values at $x = \frac{1}{2}$.
4. In your solution, display the shared coarse field and a more finely sampled field for (B), together with the corresponding curves. Explain why a finite sampled field cannot determine an arbitrary smooth differential equation uniquely.

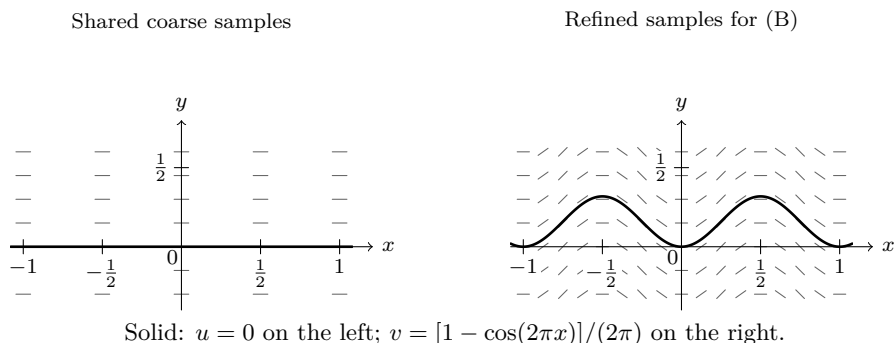
Question 9 – Solution

Strategy. Evaluate the law at the actual sampling locations before trusting a plot. A grid can systematically miss variation between columns.

Step 1: Hidden variation. Every sampled column has $x = j/2$ for an integer j . Therefore $\sin(2\pi x) = \sin(j\pi) = 0$. Both programs draw only horizontal segments there, regardless of y . At the added column,

$$f_A(\tfrac{1}{4}, y) = 0, \quad f_B(\tfrac{1}{4}, y) = 1,$$

so the fields visibly differ.



Step 2: Distinct solutions. Direct differentiation gives $u' = 0$ and

$$v'(x) = \frac{2\pi \sin(2\pi x)}{2\pi} = \sin(2\pi x), \quad u(0) = v(0) = 0.$$

At the half-unit point, $u(\frac{1}{2}) = 0, \quad v(\frac{1}{2}) = 1/\pi$. Equal sampled slopes do not imply equal solution values between or even at later grid columns.

Step 3: What a refined grid proves. The extra column disproves equality of these two laws. It does not make finite sampling sufficient for an unrestricted smooth law. For any finite set of sampled columns x_j , the nonzero smooth function $\prod_j (x - x_j)^2$ vanishes at every one of them. Adding it to a right-hand side preserves those sampled slopes while changing the field elsewhere.

Question 10

A scaled temperature model uses time x and temperature y :

$$y' = T(x) - y, \quad T(x) = 2 + \sin x.$$

The prescribed ambient temperature T varies in time. A periodic response is proposed:

$$r(x) = 2 + \frac{\sin x - \cos x}{2}.$$

Tasks

1. Interpret the slope signs above and below $y = T(x)$. Compute the slopes at $(0, 1)$, $(0, 3)$, and $(\pi/2, 2)$.
2. Test whether the ambient-temperature curve itself solves the equation. Verify that r does solve it.
3. On $0 \leq x \leq 2\pi$, find the maximum and minimum of r , including their times. Compare the time of its maximum with the time of the ambient maximum.
4. Draw the field, the ambient curve, and the response in your solution. Explain why a response extremum occurs where the response meets the zero-slope curve, although that curve is not itself a solution.

Question 10 – Solution

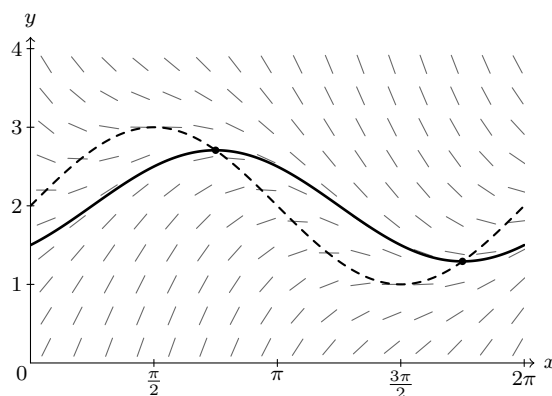
Strategy. Interpret the moving zero-slope curve as a temperature comparison, then verify the response and locate its horizontal tangents.

Step 1: Heating and cooling. Below $y = T(x)$, $y' > 0$ and the object warms; above it, $y' < 0$ and it cools. At the three listed points, the slopes are $\boxed{1, -1, 1}$.

Step 2: Check both curves. Along $y = T(x)$ the right-hand side is zero, but $T'(x) = \cos x$ is not identically zero on any interval. Thus T is not a solution. By contrast,

$$r' = \frac{\cos x + \sin x}{2} = 2 + \sin x - r,$$

so r is a solution for all real x .



Solid: response r . Dashed: ambient temperature T . Dots: response extrema.

Step 3: Extrema and delay. Rewrite

$$r(x) = 2 + \frac{1}{\sqrt{2}} \sin(x - \pi/4).$$

On the stated interval, its extreme values are

$$\boxed{r_{\max} = 2 + \frac{1}{\sqrt{2}} \text{ at } x = \frac{3\pi}{4}, \quad r_{\min} = 2 - \frac{1}{\sqrt{2}} \text{ at } x = \frac{7\pi}{4}.$$

The endpoint values both equal $3/2$ and lie between these extremes. The ambient maximum occurs at $x = \pi/2$, so the response maximum follows it by $\boxed{\pi/4}$ time units.

Step 4: Field interpretation. At a response extremum, $r' = T - r = 0$, so the response meets the zero-slope curve. Elsewhere it need not equal T . A locus of zero assigned slopes can move with time without being a solution curve itself.

Question 1

Consider the initial value problem

$$y' = e^{-x^2}, \quad y(0) = 1.$$

A proposed answer is

$$F(x) = 1 + \int_0^x e^{-t^2} dt.$$

A student rejects this expression because the integral has no elementary antiderivative. You may take that non-elementarity as given; proving it is not part of this problem.

Tasks

1. Verify F as an exact solution and state its interval of definition.
2. Prove uniqueness directly: if G is another solution with $G(0) = 1$ on an interval containing 0, show that $G = F$ there.
3. Without evaluating the integral numerically, prove the bounds

$$1 + e^{-1} < F(1) < 2.$$

4. Explain the distinction between an exact representation, an elementary formula, and a numerical approximation. Does the lack of an elementary antiderivative imply that the IVP has no solution?

Question 1 – Solution

Strategy. A definite integral defines a function. The Fundamental Theorem of Calculus can verify that function even when elementary antidifferentiation is unavailable.

Step 1: Verification and domain. The integrand is continuous for every real t , so the integral exists for every finite real x . The Fundamental Theorem gives

$$F'(x) = e^{-x^2}, \quad F(0) = 1 + 0 = 1.$$

Thus $\boxed{F \text{ is an exact solution on } \mathbb{R}.}$ Negative x causes no problem: the integral is interpreted with reversed limits.

Step 2: Uniqueness without a general theorem. If G solves the same IVP on an interval I containing 0, then

$$(G - F)' = e^{-x^2} - e^{-x^2} = 0.$$

A differentiable function with zero derivative on an interval is constant there. At 0, $G - F = 0$, so

$$\boxed{G(x) = F(x) \quad (x \in I).}$$

Step 3: Strict integral bounds. For $0 < t < 1$,

$$e^{-1} < e^{-t^2} < 1.$$

These strict inequalities hold throughout an interval of positive length. Integrating from 0 to 1 therefore gives

$$e^{-1} < \int_0^1 e^{-t^2} dt < 1, \quad \boxed{1 + e^{-1} < F(1) < 2.}$$

Step 4: What counts as a solution. The definite-integral expression specifies F exactly, with no rounding. An elementary formula is a more restricted type of expression, assembled from familiar elementary functions. A decimal obtained by numerical integration is an approximation to a value of F . Failure to find an elementary expression does not negate the exact solution already verified.

Question 2

The exact solution of $y' = y$, $y(0) = 1$, is supplied as $y = e^x$. A polynomial approximation is

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}.$$

You may use Taylor's theorem with a fourth-derivative remainder.

Tasks

1. Calculate the residual $P' - P$ and determine whether P solves the IVP exactly on any open interval containing 0.
2. Let $E(x) = e^x - P(x)$. Prove that for $0 \leq x \leq 1$,

$$\frac{x^4}{24} \leq E(x) \leq \frac{ex^4}{24}.$$

3. Give exact lower and upper bounds for $E(1)$, and identify whether P overestimates or underestimates the solution for $0 < x \leq 1$.
4. Plot the error and its two bounds in your solution. Explain why matching the initial value and several derivatives at 0 does not make the polynomial an exact interval solution.

Question 2 – Solution

Strategy. Use a residual to test exactness and Taylor's theorem to assess the approximation. These answer different questions.

Step 1: Exactness test. Since $P' = 1 + x + x^2/2$,

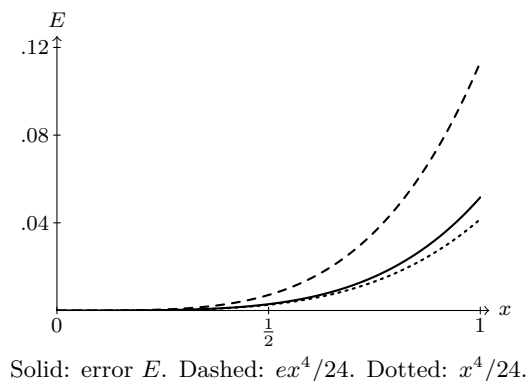
$$P' - P = -\frac{x^3}{6}.$$

The residual is nonzero whenever $x \neq 0$. Although $P(0) = 1$, the polynomial does not solve the equation on any open interval containing 0.

Step 2: Error bounds. For $0 < x \leq 1$, Taylor's theorem supplies a number ξ with $0 < \xi < x$ such that

$$E(x) = \frac{e^\xi x^4}{24}.$$

Because $1 \leq e^\xi \leq e$, the requested bounds follow; at $x = 0$ all three expressions are zero.



Step 3: Size and sign. At $x = 1$, $P(1) = 8/3$, so

$$\frac{1}{24} \leq E(1) = e - \frac{8}{3} \leq \frac{e}{24}.$$

For $x > 0$ the lower bound is positive, so P underestimates the exact solution. The two endpoint bounds are in fact strict at 1.

Step 4: Local agreement is limited evidence. The values of P and its first three derivatives agree with those of e^x at 0. That is local Taylor information. Solving the ODE requires $P'(x) = P(x)$ at every point of an interval, which the nonzero residual disproves.

Question 3

Two measurements of an initial value lead to the candidate functions

$$y(x) = e^{2x}, \quad \tilde{y}(x) = (1 + \varepsilon)e^{2x}, \quad \varepsilon = 10^{-3},$$

for the equation $y' = 2y$. Define absolute error and relative error by

$$A(x) = |\tilde{y}(x) - y(x)|, \quad R(x) = \frac{|\tilde{y}(x) - y(x)|}{|y(x)|}.$$

Tasks

1. Verify both candidates and identify their different initial values.
2. Determine $A(x)$ and $R(x)$ for $x \geq 0$. Find the first time the absolute error reaches 0.1.
3. For the equation $z' = -2z$, compare the candidates $z = e^{-2x}$ and $\tilde{z} = (1 + \varepsilon)e^{-2x}$ in the same way.
4. Plot the two absolute errors in your solution. Explain why increasing separation here is not evidence that one IVP has multiple solutions, and why absolute and relative error tell different stories.

Question 3 – Solution

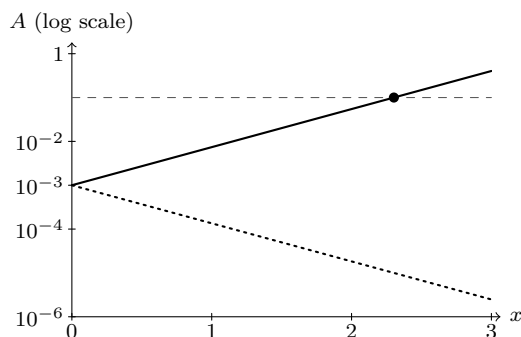
Strategy. Keep the initial conditions distinct and compute both error measures before interpreting the behavior.

Step 1: Verification. Differentiation gives $y' = 2y$ and $\tilde{y}' = 2\tilde{y}$. Their initial values are 1 and 1.001, respectively. They solve different IVPs for the same differential equation.

Step 2: Growth case. For $x \geq 0$,

$$A(x) = 10^{-3}e^{2x}, \quad R(x) = 10^{-3}.$$

The absolute error increases strictly. Its first value 0.1 occurs when $e^{2x} = 100$, giving $x = \ln 10$.



Solid: growth error. Dotted: decay error. Dashed: $A = 0.1$; dot at $x = \ln 10$.

Step 3: Decay case. Both decay candidates satisfy $z' = -2z$, again with initial values 1 and 1.001. Their errors are

$$|\tilde{z} - z| = 10^{-3}e^{-2x}, \quad \frac{|\tilde{z} - z|}{|z|} = 10^{-3}.$$

The absolute error decreases to zero and never reaches 0.1 for $x \geq 0$. The relative error stays fixed at 0.1% in both examples.

Step 4: Interpretation. Uniqueness concerns identical equations and identical initial data. Here the data differ. In the growth case the absolute discrepancy becomes large, yet it remains the same fraction of the growing reference solution. In the decay case both the solution and its discrepancy shrink proportionally.

Question 4

For the initial value problem

$$y' = y^2, \quad y(0) = 1,$$

a candidate is

$$f(x) = \frac{1}{1-x}.$$

A report claims: “The right-hand side y^2 is smooth everywhere, so this formula gives a solution through $x = 1$ and for all real x .”

Tasks

1. Verify the candidate wherever it is defined. Find the largest open interval containing 0 on which it solves the IVP.
2. Calculate its one-sided limits at $x = 1$ and explain why assigning any finite value at 1 cannot extend this solution across that point.
3. Does the same formula solve the differential equation on $(1, \infty)$? Explain why that branch is not a continuation of the given IVP solution.
4. Sketch both branches in your solution and correct the report’s inference about smooth equations and global solutions.

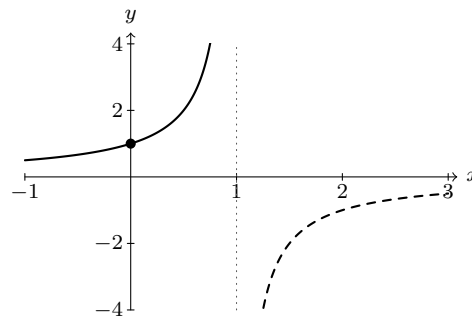
Question 4 – Solution

Strategy. Check the function's domain as well as the equation's right-hand side. A solution can become unbounded in finite time even for a smooth equation.

Step 1: Verification and interval. For $x \neq 1$,

$$f'(x) = \frac{1}{(1-x)^2} = f(x)^2, \quad f(0) = 1.$$

The largest open interval containing 0 that avoids 1 is $(-\infty, 1)$.



Solid: IVP branch. Dashed: separate solution branch. Dotted: $x = 1$.

Step 2: No finite extension. The limits are

$$\lim_{x \rightarrow 1^-} f(x) = +\infty, \quad \lim_{x \rightarrow 1^+} f(x) = -\infty.$$

A differentiable extension across 1 would be continuous there and would have a finite value. The left-hand divergence rules this out, independently of what formula is proposed after 1.

Step 3: The separate branch. On $(1, \infty)$, the same derivative calculation still proves $f' = f^2$. This branch is a solution of the differential equation, but its interval does not contain the initial point 0 and cannot be joined through $x = 1$ to the IVP branch.

Step 4: Corrected inference. Smoothness of a right-hand side is not a guarantee of existence for all time. Here the equation is defined at every finite pair (x, y) , but the IVP solution escapes to unbounded y as $x \rightarrow 1^-$. The failure belongs to global continuation of the solution, not to a singularity of y^2 at a finite point.

Question 5

An experiment records the complete trajectory $p(x) = e^{-x}$ for all $x \geq 0$, with $p(0) = 1$. Two proposed models are

$$(A) \quad y' = -y, \quad (B) \quad y' = -y + (y - e^{-x})^2.$$

Tasks

1. Verify that the recorded trajectory satisfies both models.
2. Compare the slopes predicted by the models at $(0, 0)$ and $(0, 2)$. Are the right-hand sides the same function of (x, y) ?
3. Test whether $y \equiv 0$ is an equilibrium solution of each model.
4. Explain why even a complete, noiseless trajectory does not identify an unrestricted differential equation uniquely. Generalize the construction by replacing the added square with a term involving an arbitrary smooth function $H(x, y)$.

Question 5 – Solution

Strategy. A trajectory tests a model only along one curve in the (x, y) -plane. Compare predictions away from that curve.

Step 1: Agreement on the observed curve. Since $p' = -e^{-x} = -p$, model (A) holds. Along $y = p(x)$, the added term in (B) is $(p - e^{-x})^2 = 0$, so (B) also gives $p' = -p$. Both match the entire observation and its initial value exactly.

Step 2: Different off-curve predictions. At $x = 0$, $e^{-x} = 1$. Therefore

Point	Slope from (A)	Slope from (B)
$(0, 0)$	0	1
$(0, 2)$	-2	-1

Thus the models agree on p but are different laws.

Step 3: An equilibrium distinguishes them. For $y \equiv 0$, the derivative is zero. Model (A) assigns zero, so this is an equilibrium. Model (B) assigns $e^{-2x} > 0$, so the same constant function is not a solution of (B).

Step 4: General non-identifiability. For any smooth H , the model

$$y' = -y + H(x, y)(y - e^{-x})^2$$

admits $p = e^{-x}$, because the added term vanishes identically along that curve. Many choices of H change slopes elsewhere. Additional experiments with different initial data or restrictions on the allowed model class are needed to distinguish such laws. Complete data on one trajectory constrain a curve, not the whole plane.

Question 6

Compare an equation with its squared version:

$$(A) \quad y' = -\sqrt{1-y^2}, \quad |y| \leq 1,$$

$$(B) \quad (y')^2 = 1-y^2.$$

Both are assigned $y(0) = 0$. Consider $u(x) = \sin x$ and $v(x) = -\sin x$.

Tasks

1. Verify that both candidates satisfy (B) and the initial value. Which satisfies (A) on $(-\pi/2, \pi/2)$?
2. State the sign restriction lost by squaring. Give a condition that, together with (B), recovers (A) for real differentiable functions.
3. Explain why $v = -\sin x$ does not satisfy (A) on all of $\mathbb{R}$.
4. Extend the accepted middle branch to a C^1 solution of (A) on all of $\mathbb{R}$ by adjoining constant pieces outside $[-\pi/2, \pi/2]$. Verify the joins and sketch the extension in your solution.

Question 6 – Solution

Strategy. Squaring forgets a sign. Restore that sign when selecting a branch, and check differentiability when extending it.

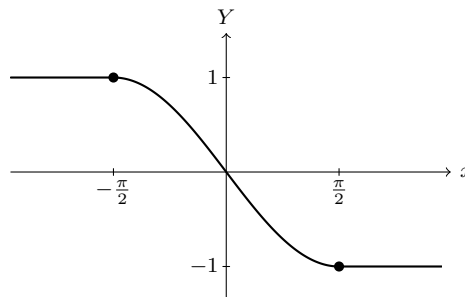
Step 1: Compare the candidates. Both satisfy $(y')^2 = \cos^2 x = 1 - y^2$ and have value zero at 0. On $(-\pi/2, \pi/2)$, $\cos x > 0$, so the right-hand side of (A) is $-|\cos x| = -\cos x$. Hence

$$v = -\sin x \text{ satisfies (A), but } u = \sin x \text{ does not.}$$

Step 2: Recover the lost information. Equation (A) forces $y' \leq 0$. Conversely, (B) and $y' \leq 0$ imply $|y| \leq 1$ and $y' = -\sqrt{1 - y^2}$. Outside the middle interval, $v' = -\cos x$ is positive wherever $\cos x < 0$, so v fails (A) there.

Step 3: A global extension. Define

$$Y(x) = \begin{cases} 1, & x \leq -\pi/2, \\ -\sin x, & -\pi/2 < x < \pi/2, \\ -1, & x \geq \pi/2. \end{cases}$$



The constant pieces meet $-\sin x$ with matching value and zero derivative.

At either join, the middle value matches the adjoining constant and its derivative $-\cos x$ tends to zero, the derivative of the constant piece. Thus Y is C^1 . On the constant pieces and at the joins, both sides of (A) are zero. On the middle piece, the verification above applies. Therefore Y solves the original IVP on $\mathbb{R}$; extension is possible here because both endpoint values and derivatives remain finite.

Question 7

Consider the IVP $y' = 0$, $y(0) = 0$ and the functions

$$u(x) = 0, \quad v(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x^2}, & x > 0. \end{cases}$$

You may use the fact that v is infinitely differentiable on $\mathbb{R}$ and that $v^{(n)}(0) = 0$ for every integer $n \geq 0$.

Tasks

1. Determine the Taylor series at 0 for both functions, using the given derivative information.
2. Calculate $v'(x)$ for $x > 0$ and decide whether v solves the IVP on any open interval containing 0.
3. Prove that u is the only solution of the IVP on an interval containing 0.
4. Explain why agreement of every derivative at a single point does not contradict your conclusion. Distinguish a smooth function from one known to equal its Taylor series near that point.

Question 7 – Solution

Strategy. Use interval verification for the ODE. Infinite-order agreement at one point is still pointwise information unless convergence to the function is established.

Step 1: Identical Taylor coefficients. For both functions, every derivative at zero is zero, so both Taylor series are

$$\sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

This describes the series built from the derivatives; it does not yet assert that the series represents either function away from zero.

Step 2: The equation fails to the right. For $x > 0$, the chain rule gives

$$v'(x) = \frac{2}{x^3} e^{-1/x^2} > 0.$$

Every open interval containing 0 includes positive x , where the required identity $v' = 0$ fails. Thus v does not solve this IVP on such an interval, despite satisfying the initial value and having every derivative there equal to those of u .

Step 3: Uniqueness directly. If $y' = 0$ on an interval, the Mean Value Theorem implies that y is constant on that interval. The initial value forces that constant to be zero. Hence $y \equiv 0$ is the unique IVP solution.

Step 4: Smoothness versus a series representation. Smoothness means the derivatives of all orders exist and are continuous. A function is analytic at a point if it equals its Taylor series in some neighborhood of that point. The supplied v is smooth but not analytic at 0: its Taylor series is zero, whereas $v(x) > 0$ for every $x > 0$. Agreement of derivatives at a single point therefore does not replace the interval identity required by the ODE.

Question 8

An implicit description of a function is proposed for $x \geq 0$:

$$y^3 + xy = 1, \quad y > 0.$$

The associated initial value problem is

$$y' = -\frac{y}{x + 3y^2}, \quad y(0) = 1.$$

You may use implicit differentiation where the coefficient of y' is nonzero.

Tasks

1. Show that for every fixed $x \geq 0$ the implicit relation has exactly one positive root y .
2. Differentiate the relation, verify the ODE and initial value, and calculate the initial slope. Explain why the denominator stays nonzero on this branch.
3. Prove that the branch decreases for $x \geq 0$ and tends to zero as $x \rightarrow \infty$.
4. Prove $2/3 < y(1) < 3/4$ and sketch the positive branch in your solution. Explain why an explicit formula obtained by solving the cubic is unnecessary for these conclusions.

Question 8 – Solution

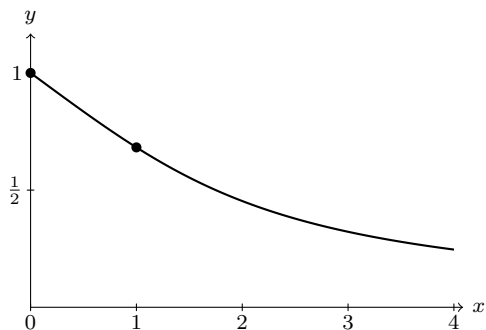
Strategy. Establish a unique branch before differentiating. Monotonicity and inequalities can describe an implicit solution without an explicit cubic formula.

Step 1: A well-defined positive branch. For fixed $x \geq 0$, let $H(y) = y^3 + xy$. It is continuous, starts at $H(0) = 0$, and tends to infinity as $y \rightarrow \infty$. For $y > 0$, $H'(y) = 3y^2 + x > 0$, so the Intermediate Value Theorem and strict increase give exactly one positive root of $H(y) = 1$.

Step 2: Verification. Differentiation yields

$$3y^2y' + y + xy' = 0, \quad \boxed{y' = -\frac{y}{x + 3y^2}}.$$

The denominator is positive on the branch, allowing local differentiable continuation, including near $(0, 1)$. At zero the relation gives $y^3 = 1$, so $y(0) = 1$ and $\boxed{y'(0) = -1/3}$.



The positive branch of $y^3 + xy = 1$; dots at $x = 0$ and $x = 1$.

Step 3: Decrease and limit. Positivity of y and the denominator implies $y' < 0$. Also $y^3 + xy = 1$ gives $0 < y < 1/x$ for $x > 0$, so the squeeze principle gives $\boxed{y(x) \rightarrow 0 \text{ as } x \rightarrow \infty}$.

Step 4: A rigorous bracket. At $x = 1$,

$$(2/3)^3 + 2/3 = 26/27 < 1, \quad (3/4)^3 + 3/4 = 75/64 > 1.$$

Strict increase in y proves $\boxed{2/3 < y(1) < 3/4}$. Root existence, differentiation and inequalities establish these results directly; an explicit cubic formula adds no needed information.

Question 9

A tank initially holds 10 liters of water containing 30 grams of salt. Fresh water enters at 2 liters per minute, and a well-mixed solution leaves at 4 liters per minute. Before the tank empties, the model is

$$V(t) = 10 - 2t, \quad S'(t) = -\frac{4S(t)}{10 - 2t}, \quad S(0) = 30,$$

where S is salt mass in grams. A candidate is

$$S(t) = 30(1 - t/5)^2.$$

Tasks

1. Explain the units and physical meaning of the right-hand side of the salt equation. Verify the candidate wherever that equation is defined.
2. State the largest mathematical solution interval containing 0, and the physical time interval for the stated tank model.
3. Find the concentration $C(t) = S(t)/V(t)$ before emptying and the limits of S and C as the emptying time is approached.
4. Plot salt mass on the physical interval in your solution. Explain why the polynomial formula does not validate continued operation of this model after the tank empties, and distinguish a concentration limit from a concentration assigned to an empty tank.

Question 9 – Solution

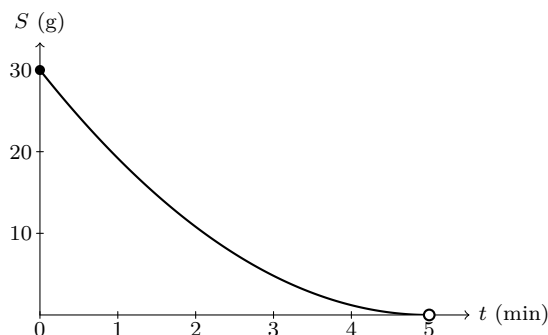
Strategy. Separate algebraic verification from the equation's domain and from the physical assumptions behind it.

Step 1: Units and substitution. The concentration S/V has units g/L; multiplying by the outflow 4 L/min gives salt loss in g/min. Fresh inflow adds no salt. The candidate satisfies

$$S' = -12(1 - t/5), \quad -\frac{4S}{10 - 2t} = -12(1 - t/5) \quad (t \neq 5),$$

and $S(0) = 30$.

Step 2: Two different intervals. The differential equation is undefined at $t = 5$. The largest open mathematical interval containing the initial time is $(-\infty, 5)$. The stated experiment starts at 0 and requires positive volume, so its physical interval is $0 \leq t < 5$ minutes.



Solid: salt mass for $0 \leq t < 5$. Open endpoint: emptying time is excluded.

Step 3: Concentration and endpoint limits. Before emptying,

$$C(t) = 3(1 - t/5) \text{ g/L}, \quad \lim_{t \rightarrow 5^-} S(t) = 0, \quad \lim_{t \rightarrow 5^-} C(t) = 0.$$

At $t = 5$, both S and V have zero limits. The quotient S/V is not defined for the empty tank, even though the pre-emptying concentration has a limit.

Step 4: Why a polynomial is not a license to extrapolate. The polynomial can be evaluated after 5, but $V(t)$ would then be negative and the assumed outflow could not be maintained from this empty tank. The algebraic formula is not sufficient to preserve the model's physical meaning or extend the original ODE across its undefined point.

Question 10

A population model in the original units is

$$\frac{dP}{dt} = \frac{1}{5}P(1 - P/50), \quad P(0) = 10,$$

where t is measured in days and 50 has the same population units as P . Define dimensionless variables

$$\tau = t/5, \quad u(\tau) = P(5\tau)/50.$$

A candidate for a normalized model is

$$u(\tau) = \frac{1}{1 + 4e^{-\tau}}.$$

Tasks

1. Use the chain rule to derive the equation and initial value for u .
2. Verify the candidate in the normalized variables, then express the corresponding $P(t)$ in the original variables.
3. Find when the population first reaches 25, giving both dimensionless time and time in days.
4. Explain why changing units alters numerical slopes and time labels without producing a different physical prediction. Identify what would go wrong if the factor from $t = 5\tau$ were omitted in differentiation.

Question 10 – Solution

Strategy. Transform both the dependent variable and time. The chain-rule factor is essential when comparing numerical rates in different units.

Step 1: Transform the IVP. Since $u(\tau) = P(5\tau)/50$,

$$\frac{du}{d\tau} = \frac{5}{50}P'(5\tau) = \frac{5}{50} \cdot \frac{1}{5}(50u)(1-u) = u(1-u).$$

The initial value is $u(0) = 10/50 = 1/5$. Hence

$$u' = u(1-u), \quad u(0) = 1/5.$$

Step 2: Verify and return to original units. The supplied function satisfies

$$u' = \frac{4e^{-\tau}}{(1+4e^{-\tau})^2} = u(1-u), \quad u(0) = 1/5.$$

Substituting $\tau = t/5$ gives

$$P(t) = \frac{50}{1+4e^{-t/5}}.$$

As a direct check, $P' = 40e^{-t/5}/(1+4e^{-t/5})^2$ agrees with the original right-hand side and $P(0) = 10$.

Step 3: First passage to half capacity. The candidate increases strictly and reaches $P = 25$ when $u = 1/2$. Therefore

$$1+4e^{-\tau} = 2 \implies \tau = \ln 4, \quad t = 5 \ln 4 \text{ days.}$$

Step 4: Same prediction, different labels. One unit of τ is five days, and one unit of u is fifty population units. Omitting the factor 5 in $du/d\tau = (5/50)P'(5\tau)$ would incorrectly give $u' = u(1-u)/5$ and misrepresent the normalized time scale. A change of units must transform rates as well as values.