

Question 1

Let S be the sphere $x^2 + y^2 + z^2 = 4$, oriented outward, and let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$.
2. Use the Divergence Theorem to find the outward flux through S .
3. Verify the result directly from the geometry of the sphere.

Question 1 – Solution

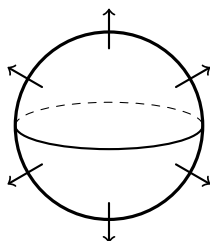
Strategy. Replace the closed-surface flux by a volume integral over the radius-2 ball.

Step 1: Divergence

$$\boxed{\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3}.$$

If E is the ball bounded by S , then

$$\text{vol}(E) = \frac{4}{3}\pi(2^3) = \frac{32\pi}{3}.$$



outward flux through $r = 2$

Step 2: Apply the theorem

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E 3 \, dV = 3 \left(\frac{32\pi}{3} \right) = 32\pi}.$$

Step 3: Direct check On the sphere, the outward unit normal is $\mathbf{n} = \langle x, y, z \rangle / 2$. Therefore

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2 + z^2}{2} = 2.$$

Multiplying by the surface area $4\pi(2^2) = 16\pi$ gives $2(16\pi) = 32\pi$.

Verification The radial field points outward everywhere, so the positive sign is required.

Question 2

Let $E = [0, 1] \times [0, 2] \times [0, 3]$ and let $S = \partial E$ have outward orientation. For

$$\mathbf{F}(x, y, z) = \langle x^2, yz, z \rangle,$$

compute the total outward flux through the six faces.

Tasks

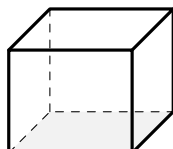
1. Find the divergence.
2. Evaluate the resulting triple integral by separating its terms.
3. Explain why this is preferable to six direct surface integrals.

Question 2 – Solution

Strategy. The box is closed and the field is smooth, so one volume integral replaces six oriented face calculations.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + z + 1.$$



$1 \times 2 \times 3$ box

Step 2: Integrate term by term The three contributions over E are

$$\begin{aligned}\iiint_E 2x \, dV &= \left(\int_0^1 2x \, dx \right) (2)(3) = 6, \\ \iiint_E z \, dV &= (1)(2) \int_0^3 z \, dz = 9, \\ \iiint_E 1 \, dV &= (1)(2)(3) = 6.\end{aligned}$$

Hence

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E (2x + z + 1) \, dV = 21.$$

Step 3: Interpret A direct method would require a different outward normal and parameter domain on every face. The divergence automatically combines all six signed contributions.

Verification The divergence is positive throughout E , so a positive net outward flux is consistent.

Question 3

Let E be the solid cylinder $x^2 + y^2 \leq 4$, $0 \leq z \leq 3$, and let $S = \partial E$ be oriented outward. For

$$\mathbf{F}(x, y, z) = \langle xz, yz, z^2 \rangle,$$

find the net outward flux and verify it by splitting S into its side and caps.

Tasks

1. Apply the Divergence Theorem in cylindrical coordinates.
2. Compute the flux through the curved side and both caps directly.
3. Compare the totals.

Question 3 – Solution

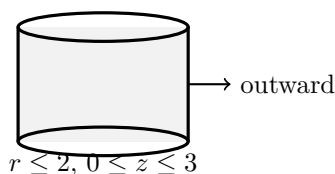
Strategy. First obtain the net flux from the divergence, then use the simple normals of a cylinder as an independent check.

Step 1: Volume integral Since

$$\nabla \cdot \mathbf{F} = z + z + 2z = 4z,$$

we obtain

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^2 \int_0^3 4z \, r \, dz \, dr \, d\theta \\ &= \boxed{72\pi}. \end{aligned}$$



Step 2: Curved side On $r = 2$, the outward vector element is $\langle 2 \cos \theta, 2 \sin \theta, 0 \rangle \, d\theta \, dz$. Thus the dot product is $4z$, and

$$\Phi_{\text{side}} = \int_0^{2\pi} \int_0^3 4z \, dz \, d\theta = 36\pi.$$

Step 3: Caps On the top $z = 3$, $\mathbf{F} \cdot \mathbf{k} = 9$, so $\Phi_{\text{top}} = 9(4\pi) = 36\pi$. On the bottom, $z = 0$, so $\Phi_{\text{bottom}} = 0$.

$$\boxed{\Phi_{\text{side}} + \Phi_{\text{top}} + \Phi_{\text{bottom}} = 72\pi}.$$

Verification The direct total matches the volume-integral result exactly.

Question 4

Let E be the spherical shell $a \leq \sqrt{x^2 + y^2 + z^2} \leq b$, where $0 < a < b$, with outward orientation on its entire boundary. For

$$\mathbf{F} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}},$$

find the net flux through the shell boundary.

Tasks

1. Compute the divergence in the shell.
2. Track the outward normal on both spherical boundary components.
3. Reconcile the separate nonzero fluxes with the net result.

Question 4 – Solution

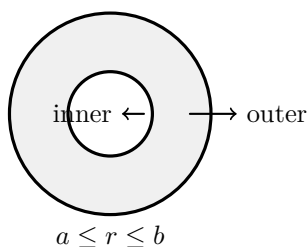
Strategy. The origin is excluded from the shell, so the field is smooth there and has zero divergence; the inner normal points toward the origin.

Step 1: Apply the theorem For $r > 0$,

$$\nabla \cdot \left(\frac{\langle x, y, z \rangle}{r^3} \right) = 0.$$

Therefore

$$\boxed{\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E 0 \, dV = 0.}$$



Step 2: Outer sphere At $r = b$, $\mathbf{F}$ points in the outward-normal direction with normal component $1/b^2$. Hence

$$\Phi_{r=b} = \frac{1}{b^2}(4\pi b^2) = 4\pi.$$

Step 3: Inner sphere The outward normal for the shell at $r = a$ points into the cavity, namely $-\mathbf{e}_r$. Thus

$$\Phi_{r=a} = -\frac{1}{a^2}(4\pi a^2) = -4\pi.$$

The two boundary components sum to zero.

Verification The cancellation depends on reversing the normal at the inner boundary; using the radial normal on both spheres would not describe the shell's outward orientation.

Question 5

Let S be the curved upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. For

$$\mathbf{F}(x, y, z) = \langle x, y, z + 1 \rangle,$$

find the flux through S by closing it with its base disk.

Tasks

1. Apply the Divergence Theorem to the closed upper half-ball.
2. Compute the flux through the base disk with its outward normal.
3. Isolate and verify the curved-surface flux.

Question 5 – Solution

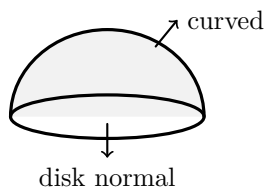
Strategy. Add the base disk, compute the total closed flux, and subtract the disk contribution with careful attention to its downward normal.

Step 1: Closed flux The divergence is

$$\nabla \cdot \mathbf{F} = 3.$$

The upper half-ball has volume $\frac{1}{2} \cdot \frac{4}{3}\pi(2^3) = 16\pi/3$. Therefore

$$\Phi_{\text{closed}} = 3 \left(\frac{16\pi}{3} \right) = 16\pi.$$



Step 2: Base disk On $z = 0$, the outward normal for the half-ball is $-\mathbf{k}$. Since $\mathbf{F} = \langle x, y, 1 \rangle$ there,

$$\Phi_{\text{disk}} = \iint_D (-1) dA = -4\pi.$$

Step 3: Curved part Because $\Phi_{\text{closed}} = \Phi_S + \Phi_{\text{disk}}$,

$$\boxed{\Phi_S = 16\pi - (-4\pi) = 20\pi}.$$

Verification On the sphere, $\mathbf{F} = \langle x, y, z \rangle + \mathbf{k}$. The radial part contributes $2(8\pi) = 16\pi$ over the hemisphere, and the constant vertical part contributes its projected area 4π , totaling 20π .

Question 6

Let E be the first-octant tetrahedron $x + y + z \leq 1$, and let $S = \partial E$ be outward oriented. For

$$\mathbf{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle,$$

compute the total outward flux.

Tasks

1. Find the divergence.
2. Evaluate the needed coordinate moments over the tetrahedron.
3. Check the answer directly on the slanted face.

Question 6 – Solution

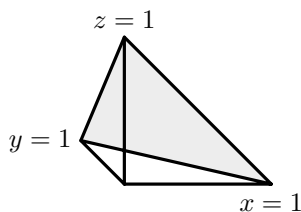
Strategy. Symmetry makes the three first moments equal; the coordinate-plane faces also provide a short direct check.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = 2x + 2y + 2z.$$

The tetrahedron has volume $1/6$ and centroid $(1/4, 1/4, 1/4)$, so

$$\iiint_E x \, dV = \iiint_E y \, dV = \iiint_E z \, dV = \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}.$$

**Step 2: Apply the theorem**

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = 2 \left(\frac{1}{24} + \frac{1}{24} + \frac{1}{24} \right) = \frac{1}{4}.}$$

Step 3: Direct check On each coordinate face the corresponding field component is zero, so those fluxes vanish. On $z = 1 - x - y$, the outward vector element is $\langle 1, 1, 1 \rangle dA$, giving

$$\iint_D (x^2 + y^2 + (1 - x - y)^2) \, dA = \frac{1}{4}.$$

Verification The direct slanted-face integral agrees with the divergence result and is positive, as expected in the first octant.

Question 7

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Its closed boundary is oriented outward. For

$$\mathbf{F}(x, y, z) = \langle 0, 0, z \rangle,$$

compute the net outward flux and verify the contributions from the two boundary pieces.

Tasks

1. Describe the cylindrical-coordinate bounds for E .
2. Apply the Divergence Theorem.
3. Compute the plane and paraboloid fluxes separately.

Question 7 – Solution

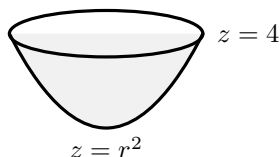
Strategy. The divergence equals 1, so the net flux is the volume; the direct check tests the downward orientation on the paraboloid.

Step 1: Bounds and volume The surfaces meet at $r = 2$, and

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad r^2 \leq z \leq 4.$$

Since $\nabla \cdot \mathbf{F} = 1$,

$$\Phi_{\text{net}} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r \, dz \, dr \, d\theta = \boxed{8\pi}.$$



Step 2: Top plane On $z = 4$, the outward normal is $\mathbf{k}$, so

$$\Phi_{\text{top}} = \iint_{r \leq 2} 4 \, dA = 16\pi.$$

Step 3: Paraboloid The solid lies above the paraboloid, so its outward vector element is downward:

$$\mathbf{n} \, dS = \langle 2x, 2y, -1 \rangle \, dA.$$

Thus

$$\Phi_{\text{paraboloid}} = \iint_{r \leq 2} -r^2 \, dA = -2\pi \int_0^2 r^3 \, dr = -8\pi.$$

The direct total is $16\pi - 8\pi = 8\pi$.

Verification The sign of the lower contribution is negative because the field points upward while the lower outward normal points downward.

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y, z) = \langle ax, 2y, 3z \rangle.$$

The measured outward flux through the unit sphere is 24π . Determine a .

Tasks

1. Express the divergence in terms of a .
2. Use the flux measurement and the unit-ball volume.
3. Verify the recovered value directly on the sphere.

Question 8 – Solution

Strategy. The divergence is constant, so the measured flux gives a linear equation for the unknown coefficient.

Step 1: Apply the theorem

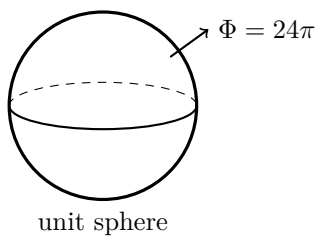
$$\nabla \cdot \mathbf{F}_a = a + 2 + 3 = a + 5.$$

The unit ball has volume $4\pi/3$, so

$$24\pi = (a + 5) \frac{4\pi}{3}.$$

Therefore $a + 5 = 18$, and

$$\boxed{a = 13}.$$



Step 2: Direct verification On the unit sphere, $\mathbf{n} = \langle x, y, z \rangle$, so with $a = 13$,

$$\mathbf{F}_{13} \cdot \mathbf{n} = 13x^2 + 2y^2 + 3z^2.$$

By spherical symmetry,

$$\iint_S x^2 dS = \iint_S y^2 dS = \iint_S z^2 dS = \frac{4\pi}{3}.$$

Hence the direct flux is

$$(13 + 2 + 3) \frac{4\pi}{3} = 24\pi.$$

Verification The recovered parameter reproduces the stated measurement by both methods.

Question 9

On $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, consider

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Let S be the outward-oriented unit sphere.

Tasks

1. Compute the flux through S directly.
2. Explain why zero divergence away from the origin does not permit applying the Divergence Theorem to the full unit ball.
3. Reconcile the result using a punctured ball.

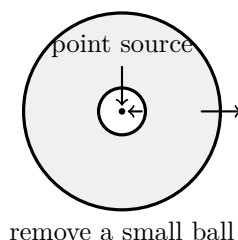
Question 9 – Solution

Strategy. The nonzero flux is caused by a singularity inside the sphere; remove a small ball so the theorem's hypotheses hold.

Step 1: Direct flux On the unit sphere, $\mathbf{F} = \mathbf{n}$, so

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S 1 \, dS = 4\pi}.$$

Although $\nabla \cdot \mathbf{F} = 0$ for $r > 0$, the field is undefined at the origin and is not smooth throughout the full ball.



Step 2: Puncture the ball Remove the ball $r < \varepsilon$. On the shell $\varepsilon \leq r \leq 1$, the divergence is zero, so the total outward flux is zero.

Step 3: Track both boundaries The outer flux is 4π . On $r = \varepsilon$, the shell's outward normal is $-\mathbf{e}_r$, while $\mathbf{F} = \mathbf{e}_r/\varepsilon^2$. Thus

$$\Phi_{r=\varepsilon} = -\frac{1}{\varepsilon^2}(4\pi\varepsilon^2) = -4\pi.$$

Therefore $4\pi - 4\pi = 0$, consistent with the theorem on the punctured region.

Verification The inner flux remains -4π as $\varepsilon \rightarrow 0^+$, recording the enclosed point source rather than disappearing.

Question 10

Let E be a bounded solid with piecewise smooth closed boundary S and outward unit normal $\mathbf{n}$.

Tasks

1. Use the Divergence Theorem with a suitable vector field to prove $\text{vol}(E) = \frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS$.
2. Apply the identity to a right circular cone of base radius R and height h , with vertex at the origin and base in the plane $z = h$.
3. Explain why the lateral surface contributes zero.

Question 10 – Solution

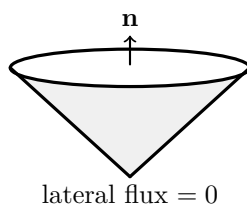
Strategy. Choose a field with divergence 1, then exploit the fact that the cone's radial position vector is tangent to its lateral generators.

Step 1: Derive the identity Let

$$\mathbf{G} = \frac{1}{3} \langle x, y, z \rangle.$$

Then $\nabla \cdot \mathbf{G} = 1$, so

$$\text{vol}(E) = \iiint_E 1 \, dV = \iint_S \mathbf{G} \cdot \mathbf{n} \, dS = \boxed{\frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS}.$$



Step 2: Lateral surface Every ray from the origin along the cone is tangent to the lateral surface. Hence $\langle x, y, z \rangle$ is tangent there and

$$\langle x, y, z \rangle \cdot \mathbf{n} = 0.$$

Step 3: Base On the base, $z = h$ and $\mathbf{n} = \mathbf{k}$, so the dot product equals h . Therefore

$$\boxed{\text{vol}(E) = \frac{1}{3} \iint_{\text{base}} h \, dA = \frac{1}{3} h (\pi R^2) = \frac{1}{3} \pi R^2 h}.$$

Verification The formula has the expected “one-third base times height” form and correct cubic units.