

Question 1

Let C be the unit circle $x^2 + y^2 = 1$ in the plane $z = 0$, oriented counterclockwise when viewed from above. For

$$\mathbf{F}(x, y, z) = \langle -y, x, z \rangle,$$

evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem.

Tasks

1. Choose an oriented spanning surface and compute $\nabla \times \mathbf{F}$.
2. Apply Stokes' Theorem.
3. Verify the answer by direct parametrization of C .

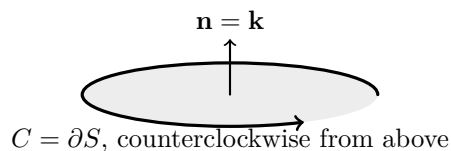
Question 1 – Solution

Strategy. Span the circle by its flat disk; the prescribed counterclockwise direction selects the upward normal.

Step 1: Curl and orientation

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & z \end{vmatrix} = \langle 0, 0, 2 \rangle.$$

Let S be the unit disk with $\mathbf{n} = \mathbf{k}$.



Step 2: Apply Stokes

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S 2 \, dA = 2\pi.$$

Step 3: Direct check With $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t).$$

Therefore $\int_0^{2\pi} \mathbf{F} \cdot \mathbf{r}' \, dt = \int_0^{2\pi} 1 \, dt = 2\pi$.

Verification The boundary is counterclockwise from the tip of $\mathbf{k}$, so the orientation pairing is correct.

Question 2

Let C be the triangular boundary of the first-octant portion of the plane $x + y + z = 1$. Orient C consistently with the normal having positive components. For

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl and an oriented vector area element.
2. Evaluate the surface integral over the projected triangle.
3. Verify the result by integrating along the three edges.

Question 2 – Solution

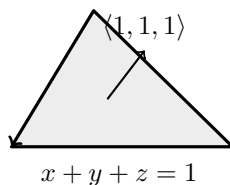
Strategy. Use the triangular plane itself; its graph vector element is constant and points in the prescribed direction.

Step 1: Curl and surface element

$$\nabla \times \mathbf{F} = \langle 1, 1, 1 \rangle.$$

Writing $z = 1 - x - y$ over $D = \{x, y \geq 0 : x + y \leq 1\}$ gives

$$\boxed{\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy}.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle dA \\ &= 3 \text{ area}(D) = \boxed{\frac{3}{2}}. \end{aligned}$$

Step 3: Edge check The orientation follows

$$(1, 0, 0) \rightarrow (0, 1, 0) \rightarrow (0, 0, 1) \rightarrow (1, 0, 0).$$

Parametrizing each edge by $0 \leq t \leq 1$ gives integrands $1 - t$, $1 - t$, and $1 - t$, respectively. Hence the three line integrals are each $1/2$, totaling $3/2$.

Verification Two successive directed edge vectors have cross product parallel to $\langle 1, 1, 1 \rangle$, confirming the boundary orientation.

Question 3

Let C be the equator of the upper hemisphere $x^2 + y^2 + z^2 = 9$, oriented as the boundary of the outward-oriented hemisphere. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ without parametrizing the hemisphere.

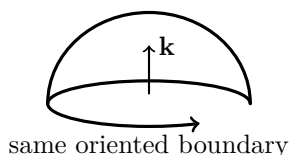
Tasks

1. Determine the induced direction of C as viewed from above.
2. Replace the hemisphere by a simpler spanning surface.
3. Evaluate the circulation and explain why the replacement is valid.

Question 3 – Solution

Strategy. Stokes' Theorem permits any smooth oriented spanning surface with the same oriented boundary; use the equatorial disk.

Step 1: Boundary orientation The outward normal on the upper hemisphere induces counterclockwise motion around the equator when viewed from $+z$.



Step 2: Choose the disk Let D be $x^2 + y^2 \leq 9$, $z = 0$, with upward normal. It has the same oriented boundary as the hemisphere. Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

Stokes' Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2(9\pi) = 18\pi.$$

Step 3: Explain the replacement Both the hemisphere and disk are orientable, share C , and lie where $\mathbf{F}$ has continuous derivatives. Each curl flux therefore equals the same boundary circulation.

Verification On the equator, $\mathbf{F} = \langle -y, x, 0 \rangle$ is tangent in the counterclockwise direction and has magnitude 3; multiplying by the circumference 6π also gives 18π .

Question 4

Let C be the intersection of the cylinder $x^2 + y^2 = 4$ with the plane $z = x + y$. Orient C consistently with the upward normal to the planar region it bounds. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Identify a convenient spanning surface and its vector element.
2. Compute the curl flux.
3. Explain why the tilt of the curve does not alter the result.

Question 4 – Solution

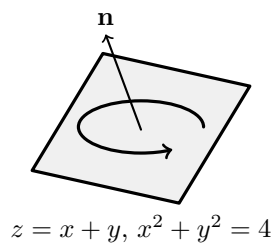
Strategy. Use the planar patch $z = x + y$ over the radius-2 disk; its vertical vector-area component is especially simple.

Step 1: Curl and vector area We have

$$\nabla \times \mathbf{F} = \langle 0, 0, 1 \rangle.$$

For $z = x + y$, the upward vector element is

$$\mathbf{n} dS = \langle -1, -1, 1 \rangle dA.$$



Step 2: Apply Stokes

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_{x^2+y^2 \leq 4} \langle 0, 0, 1 \rangle \cdot \langle -1, -1, 1 \rangle dA \\ &= \iint_{x^2+y^2 \leq 4} 1 dA = \boxed{4\pi}. \end{aligned}$$

Step 3: Interpret The curl is vertical, so only the vertical component of the oriented area vector matters. That component equals the area element of the horizontal projection, whose area is 4π .

Verification Reversing the curve would force the downward normal and change the answer to -4π .

Question 5

Let S be the annulus $1 \leq x^2 + y^2 \leq 4$ in the plane $z = 0$, oriented upward. Its boundary consists of an outer circle C_o and inner circle C_i . For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

evaluate the circulation around the fully oriented boundary ∂S .

Tasks

1. Determine the induced direction on each boundary component.
2. Apply Stokes' Theorem to the annulus.
3. Verify the result from the two separate circle integrals.

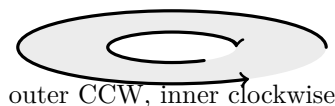
Question 5 – Solution

Strategy. For an upward annulus, the outer boundary is counterclockwise and the inner boundary is clockwise.

Step 1: Orient the boundary The right-hand rule gives

$$\partial S = C_o - C_i^{\text{CCW}}.$$

Equivalently, traverse the outer circle counterclockwise and the inner circle clockwise.



Step 2: Apply Stokes Since $\nabla \times \mathbf{F} = \mathbf{k}$,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S 1 \, dA = \pi(2^2 - 1^2) = 3\pi.$$

Step 3: Check components For a counterclockwise circle of radius R ,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \pi R^2.$$

Thus the outer circle contributes 4π , while the clockwise inner circle contributes $-\pi$. Their sum is 3π .

Verification Assigning both circles counterclockwise would give 5π and violate the induced-boundary convention.

Question 6

Use Stokes' Theorem to derive an area formula for a positively oriented simple closed curve C in the xy -plane. Then apply it to the ellipse

$$C : \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b > 0.$$

Tasks

1. Choose a vector field with constant vertical curl.
2. Derive $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
3. Evaluate the formula for the ellipse.

Question 6 – Solution

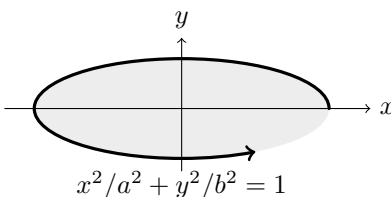
Strategy. Encode planar area as the curl flux of a field whose vertical curl is 1.

Step 1: Choose the field Let

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle.$$

Then $\nabla \times \mathbf{F} = \mathbf{k}$. If D is the region enclosed by positively oriented C , Stokes gives

$$\text{area}(D) = \iint_D 1 \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} = \boxed{\frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$



Step 2: Parametrize the ellipse Use

$$x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi.$$

Then

$$x \, dy - y \, dx = ab(\cos^2 t + \sin^2 t) \, dt = ab \, dt.$$

Therefore

$$\boxed{\text{area}(D) = \frac{1}{2} \int_0^{2\pi} ab \, dt = \pi ab}.$$

Verification Setting $a = b = R$ recovers the circle area πR^2 , including the correct scale and orientation sign.

Question 7

Let C be the unit circle in $z = 0$, counterclockwise from above. Two surfaces span it: the upper hemisphere S_1 and the paraboloid $S_2 : z = x^2 + y^2 - 1, x^2 + y^2 \leq 1$, both oriented to induce the given direction. For

$$\mathbf{F} = \langle -y, x, z \rangle,$$

compare the curl fluxes through S_1 and S_2 .

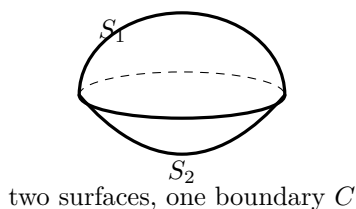
Tasks

1. Determine compatible orientations on both surfaces.
2. Evaluate one common boundary circulation or curl flux.
3. Explain why the two fluxes must agree.

Question 7 – Solution

Strategy. Use Stokes' Theorem as a surface-independence statement: each compatible curl flux equals the same circulation around C .

Step 1: Orient the surfaces Use the outward orientation on the upper hemisphere and the upward orientation on the paraboloid. Both induce counterclockwise motion on C viewed from above.



Step 2: Compute the common value Since

$$\nabla \times \mathbf{F} = \langle 0, 0, 2 \rangle,$$

we may use the upward unit disk D spanning C :

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D 2 \, dA = 2\pi.$$

Consequently,

$$\boxed{\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_1 \, dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n}_2 \, dS = 2\pi.}$$

Step 3: Explain Stokes' Theorem ties curl flux to the oriented boundary, not to the particular spanning surface, provided the field is smooth throughout the relevant surfaces.

Verification Both surfaces project once onto the unit disk with positive vertical orientation, so integrating the vertical curl 2 directly would also give 2π on each.

Question 8

Let C be the boundary of the part of the plane $z = 2x - y$ above

$$0 \leq x \leq 1, \quad 0 \leq y \leq 2,$$

oriented consistently with the normal having positive z -component. Let

$$\mathbf{F} = \frac{1}{2} \langle 2z - 3y, 3x - z, y - 2x \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Compute the curl of $\mathbf{F}$.
2. Find the oriented vector surface element for the plane.
3. Apply Stokes' Theorem and verify the orientation.

Question 8 – Solution

Strategy. The field was chosen to have constant curl, while the plane has a constant vector area element.

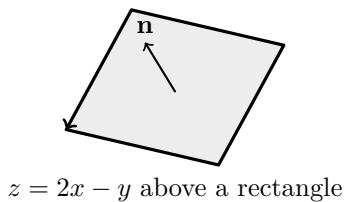
Step 1: Curl Direct differentiation gives

$$\nabla \times \mathbf{F} = \langle 1, 2, 3 \rangle.$$

Step 2: Surface element With $z = g(x, y) = 2x - y$,

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA = \langle -2, 1, 1 \rangle dA.$$

Its third component is positive, as required.



Step 3: Apply Stokes The curl flux density is constant:

$$\langle 1, 2, 3 \rangle \cdot \langle -2, 1, 1 \rangle = 3.$$

The projected rectangle has area 2, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_R 3 dA = 6.$$

Verification Reversing the traversal of all four edges would correspond to the negative normal and would change the value to -6 .

Question 9

Let S consist of the top and four side faces of the unit cube $0 \leq x, y, z \leq 1$, but not the bottom face. Give S the outward orientation. Its boundary C is the rim of the missing bottom square. For

$$\mathbf{F} = \left\langle -\frac{y}{2}, \frac{x}{2}, 0 \right\rangle,$$

compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem on the composite surface.

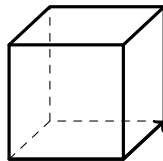
Tasks

1. Determine the induced direction of C as viewed from above.
2. Compute the curl flux through the five faces.
3. Verify the result using the missing bottom square.

Question 9 – Solution

Strategy. The curl is vertical, so the four vertical faces contribute zero; only the top face remains.

Step 1: Boundary direction The outward orientation on the four sides induces counterclockwise traversal of the bottom rim when viewed from above.



C : rim of the missing bottom

Step 2: Flux through S Since

$$\nabla \times \mathbf{F} = \mathbf{k},$$

its dot product with every horizontal side-face normal is zero. On the top, the outward normal is $\mathbf{k}$, so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = 0 + \iint_{[0,1]^2} 1 \, dA = 1.$$

Step 3: Replace the surface The upward-oriented bottom square has the same counterclockwise boundary C . Applying Stokes to that single square also gives $\iint 1 \, dA = 1$.

Verification Using the outward normal $-\mathbf{k}$ on the missing bottom face would induce the opposite boundary direction; it is not the compatible replacement for the stated C .

Question 10

On $\mathbb{R}^3$ minus the z -axis, define

$$\mathbf{F}(x, y, z) = \left\langle -\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right\rangle.$$

Let C be the unit circle in $z = 0$, counterclockwise from above.

Tasks

1. Compute $\nabla \times \mathbf{F}$ where the field is defined.
2. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly.
3. Explain why applying Stokes to the full disk is invalid, and reconcile the result using a punctured disk.

Question 10 – Solution

Strategy. The apparent contradiction “zero curl but nonzero circulation” signals a failed hypothesis: the field is singular on every full disk spanning C .

Step 1: Curl and direct integral Differentiation gives

$$\nabla \times \mathbf{F} = \mathbf{0} \quad (x^2 + y^2 > 0).$$

On C , let $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t, 0 \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = 2\pi.$$



Step 2: Locate the failed hypothesis The field is undefined at the origin, which lies on the full spanning disk. Thus $\mathbf{F}$ does not have continuous derivatives on an open set containing that disk, and Stokes' Theorem cannot be applied there.

Step 3: Use an annulus Remove a radius- ε disk. On the resulting annulus the curl is zero, so Stokes gives

$$0 = \oint_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_{\text{inner, clockwise}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi - 2\pi.$$

Verification The inner boundary carries exactly the circulation needed to account for the puncture; no contradiction remains.