

Question 1

Let S be the part of the plane $z = 1 + x + y$ above the unit square $0 \leq x, y \leq 1$, oriented upward. Let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

1. Find the upward-oriented vector surface element.
2. Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$.
3. State how the result changes under downward orientation.

Question 1 – Solution

Strategy. Parametrize the graph by (x, y) and retain the cross product itself, not merely its magnitude.

Step 1: Oriented element Use

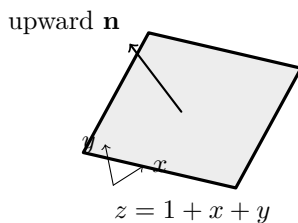
$$\mathbf{r}(x, y) = \langle x, y, 1 + x + y \rangle.$$

Then

$$\mathbf{r}_x = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_y = \langle 0, 1, 1 \rangle,$$

so the upward vector area element is

$$\mathbf{n} \, dS = (\mathbf{r}_x \times \mathbf{r}_y) \, dx \, dy = \langle -1, -1, 1 \rangle \, dx \, dy.$$



Step 2: Dot product On the surface, $z = 1 + x + y$, and hence

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = -x - y + z = 1.$$

Therefore

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^1 \int_0^1 1 \, dx \, dy = 1.$$

Step 3: Reverse orientation A downward normal is the negative of the chosen one, so the downward flux is $\boxed{-1}$.

Verification The cross product has positive third component, confirming the upward orientation; changing only the orientation must negate a flux integral.

Question 2

Let S be the paraboloid patch $z = x^2 + y^2$ above $x^2 + y^2 \leq 1$, oriented upward. For

$$\mathbf{F}(x, y, z) = \langle x, y, 1 \rangle,$$

compute the flux across S .

Tasks

1. Derive the upward vector surface element.
2. Evaluate the flux in polar coordinates.
3. Explain how positive and negative contributions cancel.

Question 2 – Solution

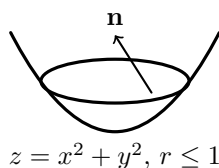
Strategy. Use the upward graph element $\langle -g_x, -g_y, 1 \rangle dA$ and then exploit radial symmetry.

Step 1: Vector area element For $g(x, y) = x^2 + y^2$,

$$\boxed{\mathbf{n} dS = \langle -2x, -2y, 1 \rangle dA}.$$

Thus

$$\mathbf{F} \cdot \mathbf{n} dS = (-2x^2 - 2y^2 + 1) dA = (1 - 2r^2) dA.$$



Step 2: Integrate

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^1 (1 - 2r^2) r dr d\theta \\ &= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^1 = \boxed{0}. \end{aligned}$$

Step 3: Locate the cancellation The normal component is positive for $r < 1/\sqrt{2}$ and negative for $r > 1/\sqrt{2}$. The inner contribution is

$$2\pi \left[\frac{r^2}{2} - \frac{r^4}{2} \right]_0^{1/\sqrt{2}} = \frac{\pi}{4},$$

while the outer annulus contributes $-\pi/4$.

Verification The exact cancellation explains why zero flux does not mean the field is tangent everywhere.

Question 3

Let S be only the curved side of the cylinder

$$x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

oriented away from the z -axis. For $\mathbf{F} = \langle x, y, z^2 \rangle$, find the outward flux.

Tasks

1. Parametrize the curved side with the correct orientation.
2. Compute the oriented vector area element.
3. Evaluate and interpret the flux.

Question 3 – Solution

Strategy. Use angle and height parameters. The outward cross product is horizontal, so the vertical component of the field contributes nothing.

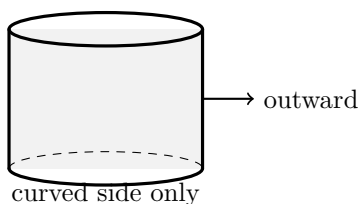
Step 1: Parametrize

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

The ordered cross product

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$$

points away from the axis.



Step 2: Form the integrand On S ,

$$\mathbf{F} = \langle 2 \cos \theta, 2 \sin \theta, z^2 \rangle,$$

and therefore

$$\mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) = 4 \cos^2 \theta + 4 \sin^2 \theta = 4.$$

Step 3: Integrate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^3 4 \, dz \, d\theta = 24\pi.$$

Verification The flux density is constantly positive, so the field points outward through every point of the curved side. The z^2 component is tangent to that side and correctly disappears.

Question 4

For $a > 0$, let S_a be the sphere $x^2 + y^2 + z^2 = a^2$, oriented outward. Consider the inverse-square radial field

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Find $\mathbf{F} \cdot \mathbf{n}$ on S_a .
2. Compute the outward flux directly.
3. Explain why the result is independent of a .

Question 4 – Solution

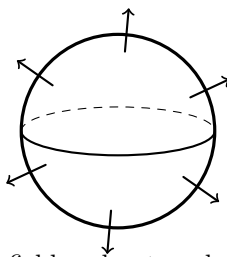
Strategy. On a centered sphere, the field and unit normal are parallel; only their magnitudes need to be compared.

Step 1: Normal component Write $\mathbf{x} = \langle x, y, z \rangle$. On S_a ,

$$\mathbf{n} = \frac{\mathbf{x}}{a}, \quad \mathbf{F} = \frac{\mathbf{x}}{a^3}.$$

Consequently,

$$\mathbf{F} \cdot \mathbf{n} = \frac{\mathbf{x} \cdot \mathbf{x}}{a^4} = \frac{1}{a^2}.$$



radial field and outward normal

Step 2: Integrate Since the sphere has area $4\pi a^2$,

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{1}{a^2} (4\pi a^2) = 4\pi.$$

Step 3: Interpret the scale Increasing the radius weakens the normal component by the factor a^{-2} while increasing spherical area by a^2 . These factors cancel exactly.

Verification The field is everywhere outward on S_a , so the positive sign is correct. The calculation is direct and uses no theorem about closed surfaces.

Question 5

Let S be the lateral cone

$$z = \sqrt{x^2 + y^2}, \quad 0 \leq z \leq 2,$$

oriented away from the z -axis. For $\mathbf{F} = \langle 0, 0, z \rangle$, compute the flux across S .

Tasks

1. Parametrize the cone and choose the requested orientation.
2. Evaluate the oriented surface integral.
3. Address the singular parameter point at the cone's vertex.

Question 5 – Solution

Strategy. The normal pointing horizontally away from the axis also has a downward vertical component, which determines the sign of the flux.

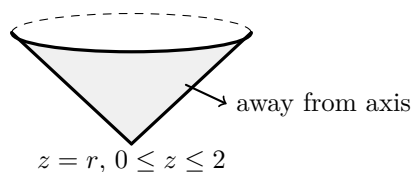
Step 1: Oriented cross product Use

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

The ordered product

$$\mathbf{r}_v \times \mathbf{r}_u = \langle u \cos v, u \sin v, -u \rangle$$

has a radial component pointing away from the axis.



Step 2: Integrate On the cone, $\mathbf{F} = \langle 0, 0, u \rangle$, so

$$\mathbf{F} \cdot (\mathbf{r}_v \times \mathbf{r}_u) = -u^2.$$

Therefore

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} \, dS &= \int_0^{2\pi} \int_0^2 -u^2 \, du \, dv \\ &= \boxed{-\frac{16\pi}{3}}. \end{aligned}$$

Step 3: Vertex The parametrization is singular at $u = 0$, but that single surface point has zero area. Equivalently, integrate over $\varepsilon \leq u \leq 2$ and let $\varepsilon \rightarrow 0^+$.

Verification The chosen normal has negative z -component while the field points upward, so a negative flux is required.

Question 6

Let E be the tetrahedron in the first octant bounded by the coordinate planes and $x + y + z = 1$. Let $S = \partial E$ carry the outward orientation, and let

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Compute the total outward flux directly by summing the four faces.

Tasks

1. Determine the flux through the three coordinate-plane faces.
2. Parametrize the slanted face with its outward orientation.
3. Sum the face contributions and verify the sign.

Question 6 – Solution

Strategy. The field has zero normal component on each coordinate face, leaving only the slanted triangular face.

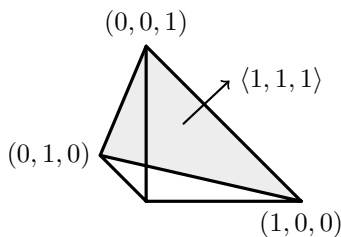
Step 1: Coordinate faces On $x = 0$, the outward normal is $-\mathbf{i}$ and $\mathbf{F} \cdot (-\mathbf{i}) = -x = 0$. Similarly, the flux densities on $y = 0$ and $z = 0$ vanish. These three fluxes are all zero.

Step 2: Slanted face Write $z = 1 - x - y$ over

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}.$$

The outward side points away from the origin, and its vector element is

$$\mathbf{n} dS = \langle 1, 1, 1 \rangle dx dy.$$



On this face,

$$\mathbf{F} \cdot \langle 1, 1, 1 \rangle = x + y + z = 1.$$

Step 3: Integrate and sum

$$\iint_{\partial E} \mathbf{F} \cdot \mathbf{n} dS = 0 + \iint_D 1 dA = \boxed{\frac{1}{2}}.$$

Verification The field points away from the origin and is tangent to the coordinate faces, so all nonzero flux must leave through the slanted face and must be positive.

Question 7

Let S be the upper hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented outward. Compute the flux of the constant field $\mathbf{F} = \langle 0, 0, 1 \rangle$ across S .

Tasks

1. Express the hemisphere as a graph and find its upward vector element.
2. Evaluate the flux using projection onto the xy -plane.
3. State the result for the opposite orientation.

Question 7 – Solution

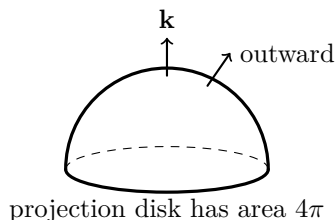
Strategy. A constant vertical field dotted with the upward graph element leaves exactly the projected area element.

Step 1: Graph orientation Write

$$z = g(x, y) = \sqrt{4 - x^2 - y^2}$$

over the disk $D : x^2 + y^2 \leq 4$. The outward orientation on the upper hemisphere is upward, so

$$\mathbf{n} dS = \langle -g_x, -g_y, 1 \rangle dA.$$



Step 2: Project Since $\mathbf{F} = \mathbf{k}$,

$$\mathbf{F} \cdot \mathbf{n} dS = \mathbf{k} \cdot \langle -g_x, -g_y, 1 \rangle dA = dA.$$

Thus

$$\boxed{\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D dA = \pi(2^2) = 4\pi}.$$

Step 3: Reverse orientation With the inward orientation the vector element changes sign, so the flux is $\boxed{-4\pi}$.

Verification The graph derivatives become unbounded at the equator, but their components are annihilated by the dot product with $\mathbf{k}$. Truncating to radius $R < 2$ and taking $R \rightarrow 2^-$ yields the same result.

Question 8

The Möbius band can be parametrized by

$$\mathbf{r}(u, v) = \left\langle \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \cos u, \left(1 + \frac{v}{2} \cos \frac{u}{2}\right) \sin u, \frac{v}{2} \sin \frac{u}{2} \right\rangle,$$

where $0 \leq u \leq 2\pi$ and $-1 \leq v \leq 1$.

Tasks

1. Verify the seam identification $\mathbf{r}(0, v) = \mathbf{r}(2\pi, -v)$.
2. Compare the transverse tangent direction at the two identified edges.
3. Explain why a globally oriented flux integral is not defined on the band.

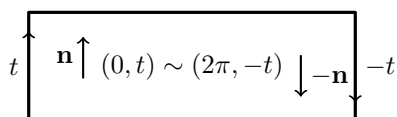
Question 8 – Solution

Strategy. Follow a normal direction once around the band and test whether it returns consistently at the identified seam.

Step 1: Identify the seam Direct substitution gives

$$\mathbf{r}(0, v) = \left\langle 1 + \frac{v}{2}, 0, 0 \right\rangle = \mathbf{r}(2\pi, -v).$$

Thus the left edge at height v is glued to the right edge at height $-v$.



the seam reverses the transverse direction

Step 2: Compare transverse tangents Along the centerline $v = 0$,

$$\mathbf{r}_v(0, 0) = \left\langle \frac{1}{2}, 0, 0 \right\rangle, \quad \mathbf{r}_v(2\pi, 0) = \left\langle -\frac{1}{2}, 0, 0 \right\rangle.$$

Also $\mathbf{r}_u(0, 0) = \mathbf{r}_u(2\pi, 0) = \langle 0, 1, 0 \rangle$. Hence

$$(\mathbf{r}_u \times \mathbf{r}_v)(0, 0) = \left\langle 0, 0, -\frac{1}{2} \right\rangle, \quad (\mathbf{r}_u \times \mathbf{r}_v)(2\pi, 0) = \left\langle 0, 0, \frac{1}{2} \right\rangle.$$

Step 3: Conclude The two parameter points represent the same surface point, yet continuation around the strip reverses the normal. Therefore no continuous global unit normal exists, and

a global oriented flux integral over the Möbius band is not defined.

Verification Cutting the band at the seam permits a local choice of normal, but gluing the seam back forces opposite choices, exposing the obstruction.

Question 9

A plane patch over the unit square has two parametrizations

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, q + 2p \rangle.$$

For $\mathbf{F} = \langle 1, 0, 0 \rangle$, compare the flux values produced by the ordered cross products of the two parametrizations.

Tasks

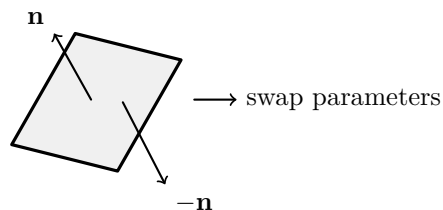
1. Show that $\mathbf{s}$ swaps the parameters of $\mathbf{r}$.
2. Compute both signed flux integrals.
3. Explain how to make the two computations represent the same geometric orientation.

Question 9 – Solution

Strategy. Swapping parameters reverses the cross product, so a vector surface integral detects a sign change that a scalar surface integral would not.

Step 1: Compare cross products Because $\mathbf{s}(p, q) = \mathbf{r}(q, p)$, the images are the same. However,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$



Step 2: Compute signed fluxes Since $\mathbf{F} = \langle 1, 0, 0 \rangle$ and both parameter domains have area 1,

$$\iint \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv = \boxed{-1},$$

whereas

$$\iint \mathbf{F} \cdot (\mathbf{s}_p \times \mathbf{s}_q) dp dq = \boxed{1}.$$

Step 3: Match orientation If the orientation from $\mathbf{r}_u \times \mathbf{r}_v$ is prescribed geometrically, then the $\mathbf{s}$ calculation must use $-(\mathbf{s}_p \times \mathbf{s}_q)$. It then also gives -1 .

Verification The two raw answers are negatives, exactly as required because the ordered normals are negatives. Flux is invariant under reparametrization only after the geometric orientation is held fixed.

Question 10

Let S be the closed cylinder $x^2 + y^2 = 1$, $0 \leq z \leq 2$, including its top and bottom disks, with outward orientation. For

$$\mathbf{F}(x, y, z) = \langle x, y, -2z \rangle,$$

compute the net outward flux directly, without using a general flux theorem.

Tasks

1. Compute the flux through the curved side.
2. Compute the flux through each cap with the correct outward normal.
3. Sum the contributions and interpret the cancellation.

Question 10 – Solution

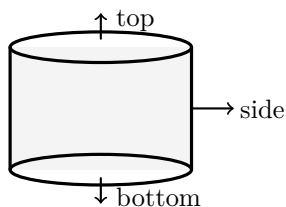
Strategy. Decompose the closed surface into three oriented pieces and track the outward normal on each one.

Step 1: Curved side Parametrize by

$$\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle.$$

The outward vector element is $\langle \cos \theta, \sin \theta, 0 \rangle d\theta dz$. Its dot product with $\mathbf{F}$ is 1, so

$$\Phi_{\text{side}} = \int_0^{2\pi} \int_0^2 1 dz d\theta = 4\pi.$$



Step 2: Caps On the top disk $z = 2$, the outward normal is $\mathbf{k}$, and

$$\mathbf{F} \cdot \mathbf{k} = -2z = -4.$$

Thus $\Phi_{\text{top}} = -4\pi$. On the bottom disk $z = 0$, the outward normal is $-\mathbf{k}$, but the vertical field component is zero, so $\Phi_{\text{bottom}} = 0$.

Step 3: Net flux

$$\boxed{\Phi_{\text{net}} = 4\pi - 4\pi + 0 = 0}.$$

Verification The side carries 4π units outward while the top carries the same amount inward; the signed balance is zero even though neither contribution vanishes.