

Question 1

Let S be the portion of the plane $z = 2x + y$ above the rectangle

$$R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}.$$

Tasks

1. Find the scalar surface element dS .
2. Compute the area of S .
3. Evaluate $\iint_S (x + z) dS$ and find the average value of $x + z$ on S .

Question 1 – Solution

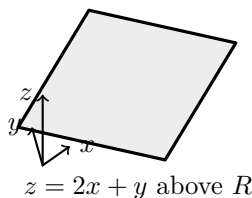
Strategy. Treat the plane as a graph, so the same constant area factor converts every integral over S into an integral over R .

Step 1: Surface element For $z = g(x, y) = 2x + y$,

$$g_x = 2, \quad g_y = 1,$$

and therefore

$$dS = \sqrt{1 + g_x^2 + g_y^2} dA = \sqrt{6} dA.$$



Step 2: Area

$$\text{area}(S) = \iint_R \sqrt{6} dA = \sqrt{6}(1)(2) = \boxed{2\sqrt{6}}.$$

Step 3: Weighted integral On S , $x + z = 3x + y$. Hence

$$\begin{aligned} \iint_S (x + z) dS &= \sqrt{6} \int_0^1 \int_0^2 (3x + y) dy dx \\ &= \sqrt{6} \int_0^1 (6x + 2) dx = \boxed{5\sqrt{6}}. \end{aligned}$$

The average value is

$$\frac{1}{\text{area}(S)} \iint_S (x + z) dS = \frac{5}{2}.$$

Verification The average $5/2$ lies between the minimum 0 and maximum 5 of $3x + y$ on R .

Question 2

Let S be the part of the paraboloid $z = x^2 + y^2$ above the unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Express dS in polar coordinates.
2. Set up and evaluate $\iint_S z \, dS$.
3. Check the radial substitution and the scale of the result.

Question 2 – Solution

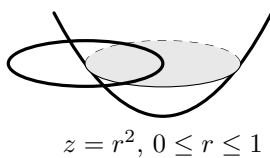
Strategy. Use the graph formula first, then exploit radial symmetry so the surface integral becomes one one-variable integral.

Step 1: Surface element With $g(x, y) = x^2 + y^2$,

$$dS = \sqrt{1 + 4x^2 + 4y^2} dA.$$

In polar coordinates, $z = r^2$ and $dA = r dr d\theta$, so

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$



$$z = r^2, 0 \leq r \leq 1$$

Step 2: Integrate

$$\iint_S z dS = 2\pi \int_0^1 r^3 \sqrt{1 + 4r^2} dr.$$

Let $t = 1 + 4r^2$. Then $r dr = dt/8$ and $r^2 = (t - 1)/4$, giving

$$\begin{aligned} \iint_S z dS &= \frac{\pi}{16} \int_1^5 (t^{3/2} - t^{1/2}) dt \\ &= \frac{\pi}{16} \left[\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2} \right]_1^5 \\ &= \boxed{\frac{\pi(25\sqrt{5} + 1)}{60}}. \end{aligned}$$

Verification The integrand is nonnegative. Also $0 \leq z \leq 1$, so the answer must not exceed the surface area; numerically it is about 2.98, consistent with that bound.

Question 3

A thin surface occupies the lateral cylinder

$$S: x^2 + y^2 = 4, \quad 0 \leq z \leq 3,$$

with surface density $\delta(x, y, z) = 1 + z$.

Tasks

1. Parametrize S and compute its surface element.
2. Find the total mass.
3. Find the center of mass, using symmetry where appropriate.

Question 3 – Solution

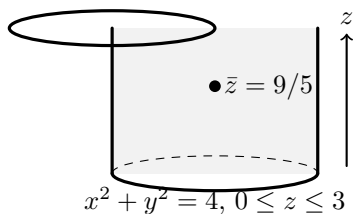
Strategy. Parametrize by angle and height. Rotational symmetry immediately locates two coordinates of the center of mass.

Step 1: Geometry

$$\mathbf{r}(\theta, z) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq z \leq 3.$$

Since $\|\mathbf{r}_\theta \times \mathbf{r}_z\| = 2$,

$$dS = 2 \, d\theta \, dz.$$



Step 2: Mass

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^3 (1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[z + \frac{z^2}{2} \right]_0^3 = \boxed{30\pi}. \end{aligned}$$

Step 3: Center of mass Symmetry gives $\bar{x} = \bar{y} = 0$. The first moment about the xy -plane is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^3 z(1+z) 2 \, dz \, d\theta \\ &= 4\pi \left[\frac{z^2}{2} + \frac{z^3}{3} \right]_0^3 = 54\pi. \end{aligned}$$

Thus

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{9}{5} \right).$$

Verification Because density increases with height, $\bar{z} = 1.8$ is correctly above the geometric midpoint 1.5 and remains inside $0 \leq z \leq 3$.

Question 4

Let S be the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 3.$$

Tasks

1. Parametrize S and derive dS .
2. Compute the surface area.
3. Evaluate $\iint_S \frac{1}{z} dS$ and its average value on S .

Question 4 – Solution

Strategy. Use height as the radius parameter. The factor of u in dS cancels against the weight $1/z$ in the second integral.

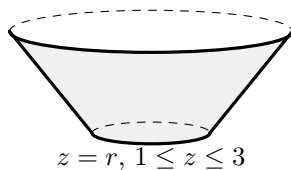
Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 3, \quad 0 \leq v \leq 2\pi.$$

The tangent vectors have cross-product magnitude

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{2},$$

so $\boxed{dS = u\sqrt{2} \, du \, dv}.$



Step 2: Area

$$\text{area}(S) = \int_0^{2\pi} \int_1^3 u\sqrt{2} \, du \, dv = \boxed{8\pi\sqrt{2}}.$$

Step 3: Weighted integral

Since $z = u$,

$$\iint_S \frac{1}{z} \, dS = \int_0^{2\pi} \int_1^3 \frac{1}{u} u\sqrt{2} \, du \, dv = \boxed{4\pi\sqrt{2}}.$$

Its average value is therefore

$$\boxed{\frac{4\pi\sqrt{2}}{8\pi\sqrt{2}} = \frac{1}{2}}.$$

Verification On the frustum, $1/3 \leq 1/z \leq 1$; the average $1/2$ lies in this interval.

Question 5

On the sphere $S : x^2 + y^2 + z^2 = 4$, evaluate the scalar surface integral

$$\iint_S z^2 dS.$$

Then find the average value of z^2 over the sphere.

Tasks

1. Choose spherical parameters and compute dS .
2. Evaluate the integral exactly.
3. Use symmetry to check the average value.

Question 5 – Solution

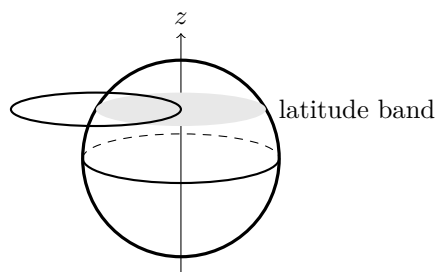
Strategy. In spherical parameters, z and dS separate into a polar factor and an azimuthal factor.

Step 1: Parametrize For radius 2, use

$$\mathbf{r}(\phi, \theta) = \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$. Then

$$z = 2 \cos \phi, \quad dS = 4 \sin \phi \, d\phi \, d\theta.$$



Step 2: Integrate

$$\begin{aligned} \iint_S z^2 \, dS &= \int_0^{2\pi} \int_0^\pi 16 \cos^2 \phi \sin \phi \, d\phi \, d\theta \\ &= 32\pi \int_0^\pi \cos^2 \phi \sin \phi \, d\phi \\ &= 32\pi \left[\frac{-\cos^3 \phi}{3} \right]_0^\pi = \boxed{\frac{64\pi}{3}}. \end{aligned}$$

Because $\text{area}(S) = 4\pi(2^2) = 16\pi$, the average is

$$\boxed{\text{Avg}_S(z^2) = \frac{4}{3}}.$$

Verification Rotational symmetry gives equal averages of x^2 , y^2 , z^2 . Their sum is always 4, so each average must be $4/3$.

Question 6

Let S be the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}, \quad z \geq 0.$$

Evaluate $\iint_S z \, dS$ by viewing S as a graph, and find the average value of z on S .

Tasks

1. Derive the graph surface element and address its boundary singularity.
2. Explain the cancellation that simplifies the integrand.
3. Evaluate the integral and average value.

Question 6 – Solution

Strategy. The graph factor becomes unbounded at the equator, but multiplication by z cancels it exactly in the interior.

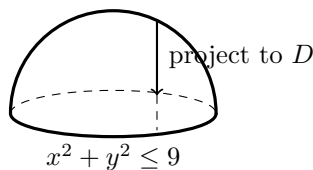
Step 1: Graph factor Let $g(x, y) = \sqrt{9 - x^2 - y^2}$ over the disk $D : x^2 + y^2 \leq 9$. Since

$$g_x = -\frac{x}{z}, \quad g_y = -\frac{y}{z},$$

we have, for $z > 0$,

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{1 + \frac{x^2 + y^2}{z^2}} = \frac{3}{z}.$$

Thus $dS = (3/z)dA$ away from the boundary.



Step 2: Cancel and integrate The weighted element is simply

$$z dS = z \frac{3}{z} dA = 3 dA.$$

Therefore

$$\boxed{\iint_S z dS = 3 \text{ area}(D) = 27\pi.}$$

The hemisphere has area $2\pi(3^2) = 18\pi$, so

$$\boxed{\text{Avg}_S(z) = \frac{27\pi}{18\pi} = \frac{3}{2}.}$$

Verification Integrating over disks of radius $R < 3$ gives $3\pi R^2$; taking $R \rightarrow 3^-$ justifies the boundary limit and yields 27π .

Question 7

Let S be the entire boundary of the unit cube $0 \leq x, y, z \leq 1$.

Tasks

1. Decompose S into its six planar faces.
2. Evaluate $\iint_S (x + y + z) dS$.
3. Find the average value of $x + y + z$ on the cube's boundary and verify it by symmetry.

Question 7 – Solution

Strategy. Pair opposite faces. Each pair produces the same contribution, avoiding six unrelated computations.

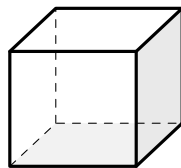
Step 1: Pair the x -faces On $x = 0$, $dS = dy dz$ and

$$\int_0^1 \int_0^1 (y + z) dy dz = 1.$$

On $x = 1$,

$$\int_0^1 \int_0^1 (1 + y + z) dy dz = 2.$$

Thus the pair perpendicular to the x -axis contributes 3.



six faces of the unit cube

Step 2: Use coordinate symmetry The cube and integrand are unchanged by permuting x, y, z . Consequently, the two y -faces and two z -faces also contribute 3 apiece. Hence

$$\boxed{\iint_S (x + y + z) dS = 3 + 3 + 3 = 9}.$$

Step 3: Average The six unit-square faces give $\text{area}(S) = 6$, so

$$\boxed{\text{Avg}_S(x + y + z) = \frac{9}{6} = \frac{3}{2}}.$$

Verification The map $(x, y, z) \mapsto (1 - x, 1 - y, 1 - z)$ preserves the boundary and pairs values whose sum is 3. Therefore their average is $3/2$, confirming the computation.

Question 8

Find the area of the helicoid patch

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Compute the tangent vectors and surface element.
2. Evaluate the resulting nonpolynomial integral.
3. Check regularity and the scale of the area.

Question 8 – Solution

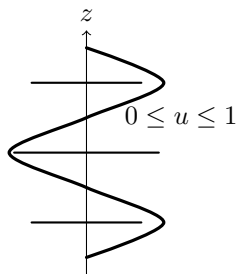
Strategy. Compute the cross product in the given parameters; its magnitude depends only on u , so the v -integration separates.

Step 1: Surface element

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle, \quad \boxed{dS = \sqrt{1 + u^2} \, du \, dv}.$$

**Step 2: Integrate**

$$\begin{aligned} \text{area}(S) &= 2\pi \int_0^1 \sqrt{1 + u^2} \, du \\ &= 2\pi \left[\frac{1}{2} \left(u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2}) \right) \right]_0^1 \\ &= \boxed{\pi \left(\sqrt{2} + \ln(1 + \sqrt{2}) \right)}. \end{aligned}$$

Step 3: Check The cross product never vanishes because its first two components have squared sum 1. Also $1 \leq \sqrt{1 + u^2} \leq \sqrt{2}$, so

$$2\pi \leq \text{area}(S) \leq 2\pi\sqrt{2},$$

and the exact result lies between these bounds.

Question 9

Let S be the entire unbounded paraboloid $z = x^2 + y^2$. Determine whether the improper scalar surface integral

$$\iint_S \frac{e^{-z}}{\sqrt{1+4z}} dS$$

converges, and evaluate it if it does.

Tasks

1. Truncate the surface above disks $x^2 + y^2 \leq R^2$.
2. Reduce the truncated surface integral to one variable.
3. Take the improper limit and justify convergence.

Question 9 – Solution

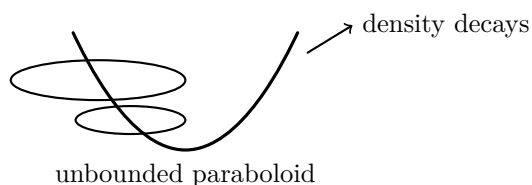
Strategy. On this paraboloid, the denominator in the density is exactly the graph-area factor, leaving a Gaussian radial integral.

Step 1: Truncate Let S_R be the part above $0 \leq r \leq R$. For $z = r^2$,

$$dS = \sqrt{1 + 4r^2} r dr d\theta.$$

Therefore the density times the area element simplifies to

$$\frac{e^{-r^2}}{\sqrt{1 + 4r^2}} \sqrt{1 + 4r^2} r dr d\theta = e^{-r^2} r dr d\theta.$$



Step 2: Evaluate the truncation

$$\begin{aligned} \iint_{S_R} \frac{e^{-z}}{\sqrt{1 + 4z}} dS &= 2\pi \int_0^R e^{-r^2} r dr \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^R \\ &= \pi(1 - e^{-R^2}). \end{aligned}$$

Step 3: Take the limit The integrand is nonnegative and the truncated values increase to a finite limit:

$$\boxed{\iint_S \frac{e^{-z}}{\sqrt{1 + 4z}} dS = \lim_{R \rightarrow \infty} \pi(1 - e^{-R^2}) = \pi.}$$

Verification The omitted tail equals πe^{-R^2} , which tends to zero and quantifies the convergence.

Question 10

The same planar patch is described over the unit square by

$$\mathbf{r}(u, v) = \langle u, v, u + 2v \rangle, \quad \mathbf{s}(p, q) = \langle 1 - p, q, 1 - p + 2q \rangle.$$

Tasks

1. Show that the parametrizations have the same image but opposite orientations.
2. Evaluate $\iint_S (x + y) \, dS$ using each parametrization.
3. Explain why reversing orientation cannot change a scalar surface integral.

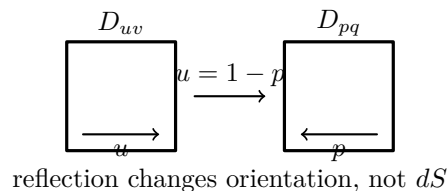
Question 10 – Solution

Strategy. Recognize $\mathbf{s}(p, q) = \mathbf{r}(1 - p, q)$, then compare cross products by magnitude rather than sign.

Step 1: Image and orientation The reflection $(u, v) = (1 - p, q)$ maps the unit square onto itself, so the images agree. Moreover,

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -1, -2, 1 \rangle, \quad \mathbf{s}_p \times \mathbf{s}_q = \langle 1, 2, -1 \rangle.$$

The normals are negatives, so the orientations are opposite; both magnitudes are $\sqrt{6}$.



Step 2: First parametrization

$$\iint_S (x + y) dS = \int_0^1 \int_0^1 (u + v) \sqrt{6} du dv = \boxed{\sqrt{6}}.$$

Step 3: Second parametrization

$$\iint_S (x + y) dS = \int_0^1 \int_0^1 (1 - p + q) \sqrt{6} dp dq = \boxed{\sqrt{6}}.$$

Verification A scalar surface integral uses $\|\mathbf{r}_u \times \mathbf{r}_v\|$, so reversing the cross product does not change dS . Here the reflection has Jacobian determinant -1 , whose absolute value is 1, confirming the equality under change of parameters.