

Question 1

Consider the surface patch

$$\mathbf{r}(u, v) = \langle u, v, 1 + 2u - v \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 1.$$

Tasks

1. Identify the surface and its parameter-domain image.
2. Find two tangent vectors and a normal vector.
3. List the four corner points of the patch.

Question 1 – Solution

Strategy. Read $x = u$ and $y = v$, eliminate the parameters, and map the rectangle's four corners.

Step 1: Identify the surface Since $x = u$ and $y = v$,

$$z = 1 + 2x - y.$$

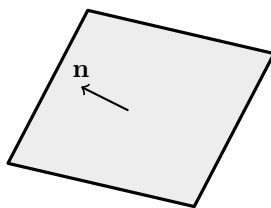
Thus the image is the portion of this plane above the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$.

Step 2: Tangent and normal vectors

$$\mathbf{r}_u = \langle 1, 0, 2 \rangle, \quad \mathbf{r}_v = \langle 0, 1, -1 \rangle.$$

Therefore

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \langle -2, 1, 1 \rangle}.$$



planar parameter rectangle

Step 3: Map the corners

$$\mathbf{r}(0, 0) = (0, 0, 1),$$

$$\mathbf{r}(2, 0) = (2, 0, 5),$$

$$\mathbf{r}(0, 1) = (0, 1, 0),$$

$$\mathbf{r}(2, 1) = (2, 1, 4).$$

These four points bound the parallelogram patch.

Verification The normal $\langle -2, 1, 1 \rangle$ is perpendicular to both tangent vectors, and each corner satisfies $z = 1 + 2x - y$.

Question 2

Parametrize the paraboloid $z = x^2 + y^2$ as a graph. At the point $P = (1, -1, 2)$, find a normal vector and the tangent plane.

Tasks

1. Give a two-parameter graph parametrization.
2. Compute the tangent vectors and their cross product at P .
3. Write and verify the tangent-plane equation.

Question 2 – Solution

Strategy. Use $x = u$, $y = v$, and let the defining graph supply the third component.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u, v, u^2 + v^2 \rangle, \quad (u, v) \in \mathbb{R}^2.$$

The point P corresponds to $(u, v) = (1, -1)$.

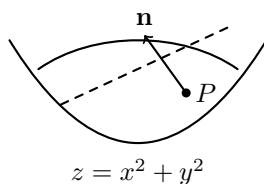
Step 2: Normal vector

$$\mathbf{r}_u = \langle 1, 0, 2u \rangle, \quad \mathbf{r}_v = \langle 0, 1, 2v \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u, -2v, 1 \rangle.$$

At P , an upward normal is $\boxed{\langle -2, 2, 1 \rangle}$.



Step 3: Tangent plane

$$-2(x - 1) + 2(y + 1) + (z - 2) = 0,$$

or

$$\boxed{z = 2x - 2y - 2}.$$

Verification Substitution of $(1, -1, 2)$ gives $2 = 2 + 2 - 2$. The normal from the graph equation $x^2 + y^2 - z = 0$ is $\langle 2, -2, -1 \rangle$, the negative of the chosen normal.

Question 3

Parametrize the half-cylinder

$$x^2 + y^2 = 4, \quad y \geq 0, \quad -1 \leq z \leq 3,$$

and choose the orientation whose normal points outward.

Tasks

1. Give a parameter domain that covers the patch once, apart from boundary seams.
2. Compute a normal from the tangent vectors.
3. Verify that the normal is outward.

Question 3 – Solution

Strategy. Use an angular parameter for the circular direction and a height parameter for z .

Step 1: Parametrize

$$\mathbf{r}(\theta, s) = \langle 2 \cos \theta, 2 \sin \theta, s \rangle, \quad 0 \leq \theta \leq \pi, \quad -1 \leq s \leq 3.$$

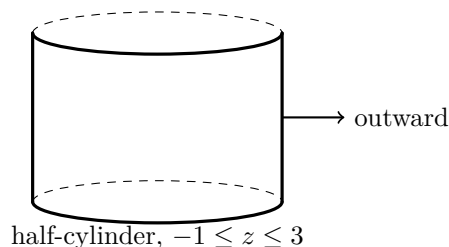
The interval $0 \leq \theta \leq \pi$ enforces $y \geq 0$.

Step 2: Tangent vectors

$$\mathbf{r}_\theta = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle, \quad \mathbf{r}_s = \langle 0, 0, 1 \rangle.$$

Their ordered cross product is

$$\boxed{\mathbf{r}_\theta \times \mathbf{r}_s = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle}.$$



Step 3: Check orientation At each point, the horizontal radial vector from the z -axis is $\langle x, y, 0 \rangle = \langle 2 \cos \theta, 2 \sin \theta, 0 \rangle$, exactly the cross product. Hence it points outward.

Verification Its magnitude is 2, never zero, so the parametrization is regular everywhere on the patch, including its boundary curves.

Question 4

Consider the standard spherical parametrization

$$\mathbf{r}(\phi, \theta) = \langle 3 \sin \phi \cos \theta, 3 \sin \phi \sin \theta, 3 \cos \phi \rangle,$$

where $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$.

Tasks

1. Show that the image is the sphere of radius 3.
2. Compute an outward normal vector from the tangent vectors.
3. Identify where the parametrization fails to be regular or one-to-one.

Question 4 – Solution

Strategy. Verify the implicit equation, then use the ordered product $\mathbf{r}_\phi \times \mathbf{r}_\theta$.

Step 1: Identify the image

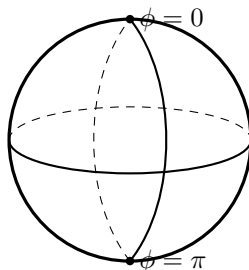
$$x^2 + y^2 + z^2 = 9 \sin^2 \phi + 9 \cos^2 \phi = 9.$$

Thus the image is $x^2 + y^2 + z^2 = 9$.

Step 2: Normal Differentiation and crossing give

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = 9 \sin \phi \langle \sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi \rangle.$$

For $0 < \phi < \pi$, this is a positive multiple of the outward radial direction.



longitude collapses at the poles

Step 3: Degeneracies Its magnitude is $9 \sin \phi$, which vanishes at $\phi = 0$ and $\phi = \pi$. Every value of θ maps to the same pole there. Also $\theta = 0$ and $\theta = 2\pi$ describe the same meridian seam.

Verification Away from the poles, $9 \sin \phi > 0$, so the tangent vectors are independent and the patch is regular locally.

Question 5

Parametrize the conical frustum

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 2,$$

and find a normal pointing away from the z -axis.

Tasks

1. Choose radial and angular parameters with correct bounds.
2. Compute both possible normal orientations.
3. Select and verify the outward normal.

Question 5 – Solution

Strategy. On this cone the radius equals the height, so one parameter can serve both roles.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u \rangle, \quad 1 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Step 2: Cross the tangents

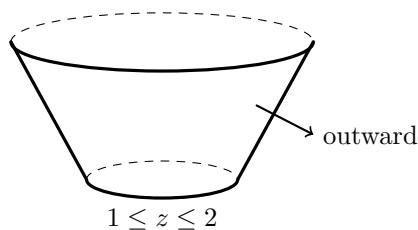
$$\mathbf{r}_u = \langle \cos v, \sin v, 1 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle.$$

Thus

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -u \cos v, -u \sin v, u \rangle.$$

This points inward horizontally. Reverse the order to obtain

$$\boxed{\mathbf{r}_v \times \mathbf{r}_u = \langle u \cos v, u \sin v, -u \rangle}.$$



Step 3: Verify Its horizontal component is $\langle x, y, 0 \rangle$, pointing away from the axis. It also agrees with the outward direction of $x^2 + y^2 - z^2 = 0$, whose gradient is $2\langle x, y, -z \rangle$.

Question 6

Analyze the parametrized surface

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 2, \quad 0 \leq v \leq 2\pi.$$

Tasks

1. Identify the surface geometrically.
2. Describe the coordinate curves $u = \text{constant}$ and $v = \text{constant}$.
3. Test regularity on the entire parameter domain.

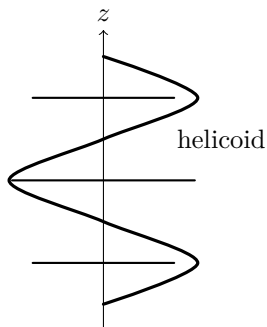
Question 6 – Solution

Strategy. Interpret radius, angle, and height directly, then test the tangent cross product.

Step 1: Identify In cylindrical terms,

$$\rho = u, \quad \theta = v, \quad z = v.$$

As the angle increases, height increases at the same rate; the image is a helicoid patch.



Step 2: Coordinate curves If $u = u_0$, then

$$\langle u_0 \cos v, u_0 \sin v, v \rangle$$

is a helix of radius u_0 ; for $u_0 = 0$ it is the z -axis. If $v = v_0$, varying u traces a radial line segment at height $z = v_0$.

Step 3: Regularity

$$\mathbf{r}_u = \langle \cos v, \sin v, 0 \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 1 \rangle,$$

and

$$\boxed{\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u \rangle}.$$

Its magnitude is $\sqrt{1 + u^2} > 0$, even at $u = 0$, so the parametrization is regular everywhere.

Verification The first two components of the cross product have squared sum 1, preventing the normal from vanishing.

Question 7

Consider

$$\mathbf{r}(u, v) = \langle u^2, v, u^3 \rangle, \quad -1 \leq u \leq 1, \quad -1 \leq v \leq 1.$$

Tasks

1. Eliminate u to find an implicit relation between x and z .
2. Determine where the parametrization is regular.
3. Explain the geometric significance of the failure set.

Question 7 – Solution

Strategy. Compare $x = u^2$ and $z = u^3$, then locate zeros of the tangent cross product.

Step 1: Eliminate the parameter

$$z^2 = u^6 = (u^2)^3 = x^3, \quad x \geq 0.$$

The image is the patch of the cuspidal cylinder $z^2 = x^3$ with $0 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Step 2: Test regularity

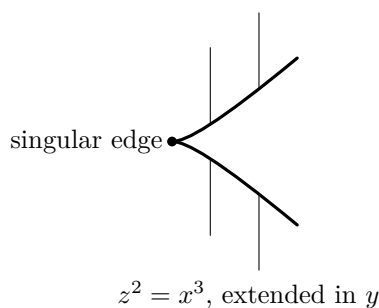
$$\mathbf{r}_u = \langle 2u, 0, 3u^2 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 0 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle -3u^2, 0, 2u \rangle.$$

This vector vanishes exactly when $u = 0$. Therefore the parametrization is regular for $u \neq 0$ and singular along

$$\boxed{\mathbf{r}(0, v) = \langle 0, v, 0 \rangle}.$$



Step 3: Interpret That line is the sharp cuspidal edge. At it, the u -direction tangent collapses to zero, so the two tangent vectors cannot span a tangent plane.

Verification For $u \neq 0$, the third component $2u$ of the cross product is nonzero, proving regularity away from the edge.

Question 8

Two parametrizations are

$$\mathbf{r}(u, v) = \langle u, v, u + v \rangle, \quad \mathbf{s}(p, q) = \langle q, p, p + q \rangle.$$

Tasks

1. Show that they have the same image.
2. Compute the ordered normal from each parametrization.
3. Determine whether they induce the same or opposite orientation.

Question 8 – Solution

Strategy. Recognize that the second parametrization swaps the two parameter roles, then compare ordered cross products.

Step 1: Compare images Both satisfy

$$x + y = z.$$

Given (u, v) for $\mathbf{r}$, choosing $(p, q) = (v, u)$ gives $\mathbf{s}(v, u) = \mathbf{r}(u, v)$. Hence their images are the same plane.

Step 2: First normal

$$\mathbf{r}_u = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_v = \langle 0, 1, 1 \rangle,$$

so

$$\mathbf{r}_u \times \mathbf{r}_v = \boxed{\langle -1, -1, 1 \rangle}.$$

Step 3: Second normal

$$\mathbf{s}_p = \langle 0, 1, 1 \rangle, \quad \mathbf{s}_q = \langle 1, 0, 1 \rangle,$$

and therefore

$$\mathbf{s}_p \times \mathbf{s}_q = \boxed{\langle 1, 1, -1 \rangle}.$$

The normals are negatives, so the parametrizations induce opposite orientations.

Verification Swapping two parameters reverses the order in a cross product, and $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

Question 9

Let

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, u^2 \rangle, \quad 1 \leq u \leq 2, \quad 0 \leq v \leq \pi.$$

Describe all boundary curves of the surface patch.

Tasks

1. Find the curves corresponding to $u = 1$ and $u = 2$.
2. Find the curves corresponding to $v = 0$ and $v = \pi$.
3. Identify the surface and verify that the four curves join correctly.

Question 9 – Solution

Strategy. Each edge of the rectangular parameter domain maps to one boundary curve.

Step 1: Constant- u edges

$$u = 1 : \langle \cos v, \sin v, 1 \rangle, \quad u = 2 : \langle 2 \cos v, 2 \sin v, 4 \rangle, \quad 0 \leq v \leq \pi.$$

These are semicircles at heights 1 and 4.

Step 2: Constant- v edges

$$v = 0 : \langle u, 0, u^2 \rangle, \quad v = \pi : \langle -u, 0, u^2 \rangle, \quad 1 \leq u \leq 2.$$

These are parabolic side curves in the plane $y = 0$.



half-annular paraboloid patch

Step 3: Identify and join Since $x^2 + y^2 = u^2$ and $z = u^2$, the patch lies on $z = x^2 + y^2$ above the half-annulus $1 \leq x^2 + y^2 \leq 4$, $y \geq 0$.

Verification The four corner images are $(1, 0, 1)$, $(-1, 0, 1)$, $(2, 0, 4)$, and $(-2, 0, 4)$; each occurs as an endpoint of exactly two adjacent boundary curves.

Question 10

A torus has major radius 3 and tube radius 1, centered about the z -axis.

Tasks

1. Construct a parametrization using two angles.
2. Compute an outward-pointing normal vector.
3. Prove the parametrization is regular everywhere and describe its seam identifications.

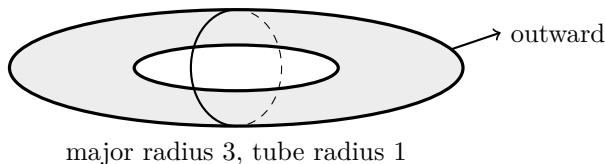
Question 10 – Solution

Strategy. First parametrize a radius-1 circle centered 3 units from the axis, then rotate it about the z -axis.

Step 1: Parametrize

$$\mathbf{r}(u, v) = \langle (3 + \cos v) \cos u, (3 + \cos v) \sin u, \sin v \rangle,$$

with $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$.



Step 2: Compute a normal Let $A = 3 + \cos v$. Then

$$\mathbf{r}_u = \langle -A \sin u, A \cos u, 0 \rangle,$$

$$\mathbf{r}_v = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle.$$

Their cross product is

$$\mathbf{r}_u \times \mathbf{r}_v = A \langle \cos u \cos v, \sin u \cos v, \sin v \rangle,$$

which points from the tube's center circle toward the surface, hence outward.

Step 3: Regularity and seams The unit vector in brackets has magnitude 1, while $A = 3 + \cos v \geq 2$. Thus the cross product never vanishes. The edges $u = 0, 2\pi$ are identified, as are $v = 0, 2\pi$.

Verification The distance from the z -axis is $3 + \cos v$ and the height is $\sin v$, so $(\sqrt{x^2 + y^2} - 3)^2 + z^2 = 1$, the intended torus.