

Question 1

For

$$\mathbf{F}(x, y, z) = \langle x^2y, yz^2, xz \rangle,$$

compute $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

Tasks

1. Compute the divergence term by term.
2. Compute all three components of the curl.
3. Evaluate both quantities at $(1, 2, -1)$.

Question 1 – Solution

Strategy. Label the components P, Q, R and apply the divergence and curl formulas in a fixed order.

Step 1: Divergence With $P = x^2y$, $Q = yz^2$, and $R = xz$,

$$\nabla \cdot \mathbf{F} = P_x + Q_y + R_z = 2xy + z^2 + x.$$

Thus

$$\boxed{\nabla \cdot \mathbf{F} = 2xy + z^2 + x}.$$

Step 2: Curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - z, 0 - x^2 \rangle.\end{aligned}$$

Therefore

$$\boxed{\nabla \times \mathbf{F} = \langle -2yz, -z, -x^2 \rangle}.$$

Step 3: Evaluate At $(1, 2, -1)$,

$$\boxed{\nabla \cdot \mathbf{F} = 6}, \quad \boxed{\nabla \times \mathbf{F} = \langle 4, 1, -1 \rangle}.$$

Verification The curl components follow the cyclic order $(R_y - Q_z, P_z - R_x, Q_x - P_y)$; recomputing from the determinant form gives the same signs.

Question 2

Consider the position field

$$\mathbf{F}(x, y, z) = \langle x, y, z \rangle.$$

Tasks

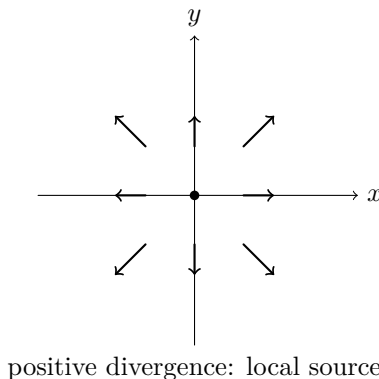
1. Compute its divergence and curl.
2. Interpret the divergence as local expansion or contraction.
3. Explain why the field has no local rotational tendency.

Question 2 – Solution

Strategy. Differentiate the matching coordinate in each divergence term and compare the cross derivatives for curl.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = \boxed{3}.$$



The positive constant divergence describes uniform local expansion: arrows point outward and grow with distance from the origin.

Step 2: Curl

Every cross derivative is zero, so

$$\nabla \times \mathbf{F} = \langle 0 - 0, 0 - 0, 0 - 0 \rangle = \boxed{\mathbf{0}}.$$

Step 3: Interpret Curl measures infinitesimal rotational tendency. This radial field expands directly away from the origin without circulating around any axis, consistent with zero curl.

Verification The field is $\nabla[(x^2 + y^2 + z^2)/2]$. A gradient field has zero curl where the second partial derivatives are continuous, confirming the calculation.

Question 3

Let

$$\mathbf{F}(x, y, z) = \langle -2y, 2x, z^2 \rangle.$$

At $P = (1, -1, 2)$, determine the local expansion rate and the curl vector.

Tasks

1. Compute and evaluate the divergence.
2. Compute and evaluate the curl.
3. Interpret the curl's axis, direction, and infinitesimal angular-speed relation.

Question 3 – Solution

Strategy. Use divergence for local volume change and curl for twice the infinitesimal angular-velocity vector.

Step 1: Expansion rate

$$\nabla \cdot \mathbf{F} = 0 + 0 + 2z = 2z.$$

At P , this is

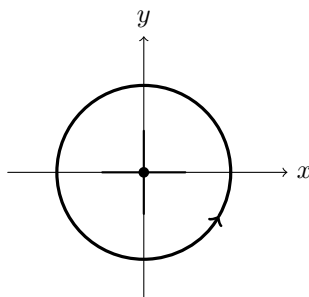
$$\boxed{(\nabla \cdot \mathbf{F})(P) = 4},$$

so the field has positive local expansion there.

Step 2: Curl

$$\nabla \times \mathbf{F} = \langle 0 - 0, 0 - 0, 2 - (-2) \rangle = \boxed{\langle 0, 0, 4 \rangle}.$$

This curl is constant, so its value at P is the same vector.



$\nabla \times \mathbf{F}$ points along $+z$

Step 3: Interpret The curl points along the positive z -axis, indicating counterclockwise rotation in horizontal planes when viewed from $+z$. The associated infinitesimal angular-velocity vector is half the curl:

$$\boxed{\vec{\omega} = \langle 0, 0, 2 \rangle}.$$

Verification The horizontal field $\langle -2y, 2x \rangle$ is rigid rotation with angular speed 2, which matches $|\nabla \times \mathbf{F}|/2 = 2$.

Question 4

For constants a, b, d , consider

$$\mathbf{F}(x, y, z) = \langle ax - by, bx - ay, dz \rangle.$$

Classify all parameter triples (a, b, d) for which the field is both divergence-free and curl-free on $\mathbb{R}^3$.

Tasks

1. Compute the divergence.
2. Compute the curl.
3. Solve the simultaneous parameter conditions and verify the resulting family.

Question 4 – Solution

Strategy. Treat the scalar divergence and all three curl components as simultaneous identities.

Step 1: Divergence

$$\nabla \cdot \mathbf{F} = a + (-a) + d = d.$$

Thus divergence-free requires $d = 0$.

Step 2: Curl With $P = ax - by$, $Q = bx - ay$, and $R = dz$,

$$\nabla \times \mathbf{F} = \langle 0, 0, Q_x - P_y \rangle = \langle 0, 0, b - (-b) \rangle = \langle 0, 0, 2b \rangle.$$

Thus curl-free requires $b = 0$.

Step 3: Classify The parameter a is unrestricted, so

$$\boxed{(a, b, d) = (a, 0, 0), \quad a \in \mathbb{R}}.$$

The resulting field is

$$\mathbf{F} = \langle ax, -ay, 0 \rangle.$$

Verification Its divergence is $a - a + 0 = 0$, and all cross derivatives vanish. Hence every field in the boxed family satisfies both conditions, including the zero field when $a = 0$.

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle xy^2, yz^2, zx^2 \rangle.$$

Verify directly that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Tasks

1. Compute $\nabla \times \mathbf{F}$.
2. Take its divergence.
3. Explain why this is an instance of a general vector identity.

Question 5 – Solution

Strategy. Compute the curl componentwise, then differentiate each component with respect to its matching coordinate.

Step 1: Curl Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle 0 - 2yz, 0 - 2xz, 0 - 2xy \rangle \\ &= \boxed{\langle -2yz, -2xz, -2xy \rangle}.\end{aligned}$$

Step 2: Divergence of the curl

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(-2yz) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(-2xy) \\ &= 0 + 0 + 0 = \boxed{0}.\end{aligned}$$

Step 3: General identity For a field with continuous second partial derivatives, expanding $\nabla \cdot (\nabla \times \mathbf{F})$ produces pairs of equal mixed partials with opposite signs. They cancel by equality of mixed partial derivatives.

Verification Each component of this particular curl omits the coordinate with respect to which divergence differentiates it, so all three terms vanish independently.

Question 6

For

$$f(x, y, z) = x^2y + yz^2 + xz,$$

verify directly that

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Tasks

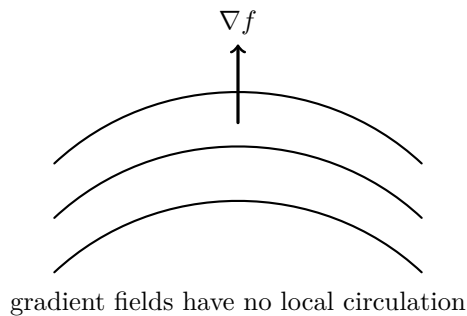
1. Compute ∇f .
2. Compute its curl and show each cancellation.
3. State the smoothness principle behind the identity.

Question 6 – Solution

Strategy. Differentiate once to obtain the gradient and again in the curl's cross-partial pairs.

Step 1: Gradient

$$\nabla f = \langle 2xy + z, x^2 + z^2, 2yz + x \rangle.$$



Step 2: Curl of the gradient Let the gradient components be P, Q, R . Then

$$R_y - Q_z = 2z - 2z = 0,$$

$$P_z - R_x = 1 - 1 = 0,$$

$$Q_x - P_y = 2x - 2x = 0.$$

Therefore

$$\boxed{\nabla \times (\nabla f) = \mathbf{0}}.$$

Step 3: General principle Each cancellation is equality of a mixed-partial pair:

$$f_{zy} = f_{yz}, \quad f_{xz} = f_{zx}, \quad f_{yx} = f_{xy}.$$

Continuous second partial derivatives guarantee these equalities.

Verification The polynomial f has continuous derivatives of every order on $\mathbb{R}^3$, so there are no domain or differentiability exceptions to the identity here.

Question 7

For the planar field

$$\mathbf{F}(x, y) = \langle x^2 - y, x + y^2 \rangle,$$

compute its planar divergence and scalar curl, then evaluate them at $(-1, 1)$.

Tasks

1. Compute $P_x + Q_y$.
2. Compute $Q_x - P_y$.
3. Interpret the two values at the specified point.

Question 7 – Solution

Strategy. In two dimensions, divergence is scalar and curl is represented by the z -component of the three-dimensional curl.

Step 1: Divergence With $P = x^2 - y$ and $Q = x + y^2$,

$$\nabla \cdot \mathbf{F} = P_x + Q_y = 2x + 2y.$$

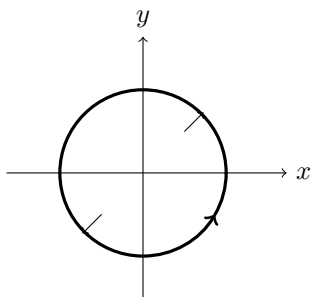
At $(-1, 1)$,

$$\boxed{\nabla \cdot \mathbf{F} = 0}.$$

Step 2: Scalar curl

$$\text{curl}_{2D} \mathbf{F} = Q_x - P_y = 1 - (-1) = \boxed{2}.$$

Equivalently, the three-dimensional embedding $\langle P, Q, 0 \rangle$ has curl $\langle 0, 0, 2 \rangle$.



zero divergence, positive scalar curl

Step 3: Interpret At the point there is no first-order net expansion or contraction, but there is positive counterclockwise rotational tendency. Zero divergence does not imply zero curl.

Verification The scalar curl is constant everywhere, while the divergence vanishes on the line $y = -x$; the point $(-1, 1)$ lies on that line.

Question 8

A linear planar field has the expansion-rotation form

$$\mathbf{F}(x, y) = \langle ax - by, bx + ay \rangle.$$

Its divergence is 4 and its scalar curl is 6 everywhere.

Tasks

1. Determine a and b .
2. Write the resulting field explicitly.
3. Separate it into pure expansion and pure rotation parts.

Question 8 – Solution

Strategy. Divergence isolates the expansion coefficient, while scalar curl isolates twice the rotation coefficient.

Step 1: Compute the invariants

$$\nabla \cdot \mathbf{F} = a + a = 2a, \quad \text{curl}_{2D} \mathbf{F} = b - (-b) = 2b.$$

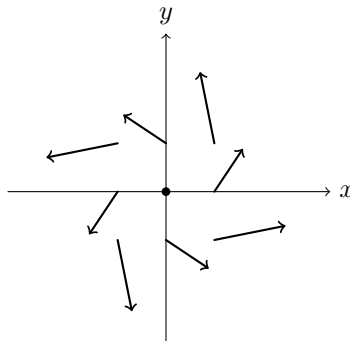
Thus

$$2a = 4, \quad 2b = 6,$$

so $\boxed{a = 2}$ and $\boxed{b = 3}$.

Step 2: Write the field

$$\boxed{\mathbf{F}(x, y) = \langle 2x - 3y, 3x + 2y \rangle}.$$



outward expansion plus counterclockwise rotation

Step 3: Decompose

$$\mathbf{F} = \underbrace{\langle 2x, 2y \rangle}_{\text{pure expansion}} + \underbrace{\langle -3y, 3x \rangle}_{\text{counterclockwise rotation}}.$$

The first part has divergence 4 and zero curl; the second has zero divergence and scalar curl 6.

Verification Recombining the two displayed parts returns both components of the boxed field and reproduces the prescribed invariants.

Question 9

On $D = \mathbb{R}^3 \setminus \{\mathbf{0}\}$, define

$$\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Compute $\nabla \cdot \mathbf{F}$ on D .
2. Compute or otherwise justify $\nabla \times \mathbf{F}$ on D .
3. Explain why the excluded origin must be stated.

Question 9 – Solution

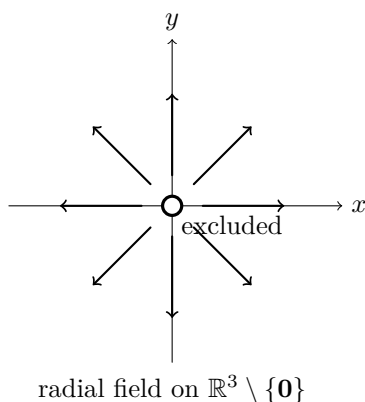
Strategy. Write $r = (x^2 + y^2 + z^2)^{1/2}$ and differentiate xr^{-3} , yr^{-3} , and zr^{-3} .

Step 1: Divergence Since

$$\frac{\partial}{\partial x}(xr^{-3}) = r^{-3} - 3x^2r^{-5},$$

with analogous formulas for y and z ,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= 3r^{-3} - 3(x^2 + y^2 + z^2)r^{-5} \\ &= 3r^{-3} - 3r^2r^{-5} = \boxed{0}, \quad r > 0.\end{aligned}$$



Step 2: Curl Because

$$\mathbf{F} = \nabla \left(-\frac{1}{r} \right)$$

on D , the curl-of-a-gradient identity gives

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0}}.$$

Step 3: Domain caveat The formula is undefined at the origin, and the derivatives used above do not exist there. The conclusions are pointwise statements on D , not statements at $\mathbf{0}$.

Verification Direct cross differentiation gives, for example, $R_y - Q_z = -3yzzr^{-5} - (-3yzzr^{-5}) = 0$, and the other two components cancel similarly.

Question 10

Let

$$\mathbf{F}(x, y, z) = \langle -yz, xz, xy \rangle.$$

Tasks

1. Compute the divergence and curl everywhere.
2. Evaluate the curl at $P = (1, 2, 3)$ and give its direction.
3. Find every point at which the curl vanishes.

Question 10 – Solution

Strategy. Compute the global formulas first; then solve the component equations of the curl simultaneously.

Step 1: Divergence Each component omits its matching differentiation variable, so

$$\nabla \cdot \mathbf{F} = \frac{\partial(-yz)}{\partial x} + \frac{\partial(xz)}{\partial y} + \frac{\partial(xy)}{\partial z} = \boxed{0}.$$

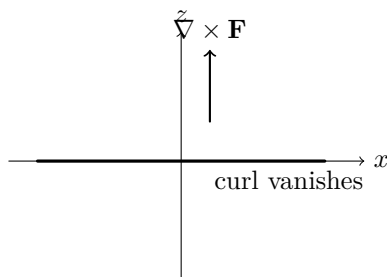
Step 2: Curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \\ &= \langle x - x, -y - y, z - (-z) \rangle \\ &= \boxed{\langle 0, -2y, 2z \rangle}.\end{aligned}$$

At $P = (1, 2, 3)$,

$$(\nabla \times \mathbf{F})(P) = \boxed{\langle 0, -4, 6 \rangle},$$

which points in the direction $\boxed{\langle 0, -2, 3 \rangle / \sqrt{13}}$.



Step 3: Vanishing locus The curl is zero exactly when

$$-2y = 0, \quad 2z = 0.$$

Thus $y = z = 0$ while x is arbitrary:

$$\boxed{\nabla \times \mathbf{F} = \mathbf{0} \text{ exactly on the } x\text{-axis}.}$$

Verification Substituting $(x, 0, 0)$ into the curl formula gives $\mathbf{0}$, and any point off the x -axis has $y \neq 0$ or $z \neq 0$, making at least one curl component nonzero.