

Question 1

Let C be the positively oriented boundary of the rectangle

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq 3\}.$$

Evaluate

$$\oint_C (x^2 - y) dx + (x + y^2) dy$$

using Green's Theorem.

Tasks

1. Identify P , Q , and the required derivative difference.
2. Convert the line integral to a double integral.
3. Evaluate and check the positive orientation.

Question 1 – Solution

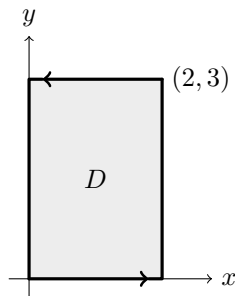
Strategy. Apply the circulation form $\oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA$.

Step 1: Differentiate Here

$$P = x^2 - y, \quad Q = x + y^2,$$

so

$$Q_x - P_y = 1 - (-1) = 2.$$



Step 2: Integrate over the region

$$\oint_C P dx + Q dy = \iint_D 2 dA = 2 \text{Area}(D) = 2(2)(3) = \boxed{12}.$$

Step 3: Orientation Positive orientation means counterclockwise traversal, with the region on the left. That is the orientation for which Green's Theorem gives the displayed positive sign.

Verification The derivative difference is the constant 2 and the rectangle has area 6, so the value must be 12; reversing C would give -12 .

Question 2

Let C be the counterclockwise ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

Evaluate

$$\frac{1}{2} \oint_C (-y \, dx + x \, dy)$$

and explain its geometric meaning.

Tasks

1. Apply Green's Theorem to the given integral.
2. Compute the enclosed area.
3. Explain the effect of reversing the ellipse's orientation.

Question 2 – Solution

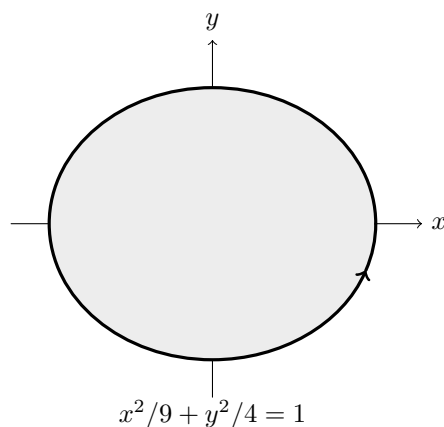
Strategy. The chosen coefficients make $Q_x - P_y = 1$, so the line integral measures area.

Step 1: Set P and Q Write

$$P = -\frac{y}{2}, \quad Q = \frac{x}{2}.$$

Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1.$$



Step 2: Apply Green's Theorem

$$\frac{1}{2} \oint_C (-y \, dx + x \, dy) = \iint_D 1 \, dA = \text{Area}(D).$$

The ellipse has semiaxes 3 and 2, so

$$\boxed{\frac{1}{2} \oint_C (-y \, dx + x \, dy) = \pi(3)(2) = 6\pi}.$$

Step 3: Reverse orientation Clockwise traversal negates the line integral, producing -6π . Geometric area itself remains 6π ; the sign records orientation.

Verification The area formula reduces to πab , and substituting $a = 3$, $b = 2$ confirms 6π .

Question 3

The planar field

$$\mathbf{F}(x, y) = \langle x^2, xy \rangle$$

acts on the disk $D : (x - 1)^2 + y^2 \leq 1$. Let $C = \partial D$ be counterclockwise. Find the outward flux of $\mathbf{F}$ across C .

Tasks

1. State the flux form of Green's Theorem.
2. Compute the divergence and its integral over D .
3. Use symmetry or the centroid of the disk to finish.

Question 3 – Solution

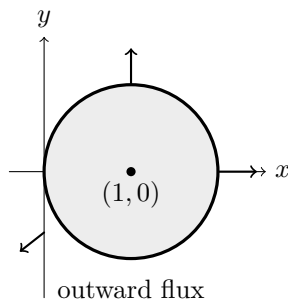
Strategy. Convert outward flux to the double integral of the planar divergence.

Step 1: Flux form For $\mathbf{F} = \langle M, N \rangle$,

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C M \, dy - N \, dx = \iint_D (M_x + N_y) \, dA.$$

Here

$$M_x + N_y = 2x + x = 3x.$$



Step 2: Integrate The disk has area π and centroid $(1, 0)$. Therefore its average x -coordinate is 1, so

$$\iint_D x \, dA = 1 \cdot \pi = \pi.$$

Consequently,

$$\boxed{\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D 3x \, dA = 3\pi.}$$

Verification The disk lies in $0 \leq x \leq 2$, so the divergence $3x$ is nonnegative. A positive outward flux is therefore consistent with the sign of the integrand.

Question 4

Let D be the annulus $1 \leq x^2 + y^2 \leq 4$ and

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

The boundary consists of the outer circle and the inner circle.

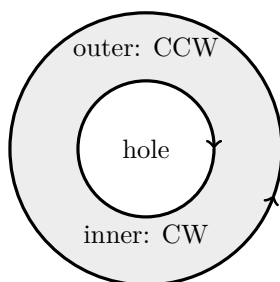
Tasks

1. Specify the positive orientation of both boundary components.
2. Apply Green's Theorem on the annulus.
3. Compute each circle's contribution and verify the cancellation.

Question 4 – Solution

Strategy. On a region with a hole, positive orientation is counterclockwise outside and clockwise around the hole.

Step 1: Orientation Traverse the radius-2 circle counterclockwise and the radius-1 circle clockwise. In both cases the annular region remains on the left.



Step 2: Apply Green's Theorem On D , the field is continuously differentiable and

$$Q_x - P_y = 0.$$

Therefore

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D 0 \, dA = \boxed{0}.$$

Step 3: Check each component On any counterclockwise circle centered at the origin, the integral is 2π , independent of radius. Hence

$$\int_{C_{\text{outer}}} \mathbf{F} \cdot d\mathbf{r} = 2\pi, \quad \int_{C_{\text{inner}}} \mathbf{F} \cdot d\mathbf{r} = -2\pi$$

because the inner circle is clockwise. Their sum is zero.

Verification If both circles were traversed counterclockwise, the total would be 4π ; that is not the positively oriented boundary of the annulus.

Question 5

Let C be the counterclockwise boundary of the region between

$$y = x^2 \quad \text{and} \quad y = x, \quad 0 \leq x \leq 1.$$

Evaluate

$$\oint_C -y \, dx + x^2 \, dy.$$

Tasks

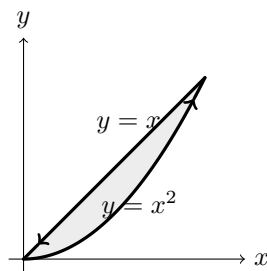
1. Describe the positively oriented boundary.
2. Set up the Green's Theorem double integral with correct bounds.
3. Evaluate and verify the sign.

Question 5 – Solution

Strategy. Replace the two curved boundary integrals with one Type I double integral over the enclosed region.

Step 1: Derivative difference Here

$$P = -y, \quad Q = x^2, \quad Q_x - P_y = 2x + 1.$$



Positive orientation travels from $(0, 0)$ to $(1, 1)$ along the lower parabola and returns along the upper line.

Step 2: Set up and evaluate

$$\begin{aligned} \oint_C P dx + Q dy &= \int_0^1 \int_{x^2}^x (2x + 1) dy dx \\ &= \int_0^1 (2x + 1)(x - x^2) dx \\ &= \int_0^1 (x + x^2 - 2x^3) dx \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \boxed{\frac{1}{3}}. \end{aligned}$$

Verification On $0 \leq x \leq 1$, both $2x + 1$ and the vertical thickness $x - x^2$ are nonnegative, so the positive result is required.

Question 6

Use a Green's Theorem area formula to find the area enclosed by the cardioid

$$r = 1 + \cos \theta, \quad 0 \leq \theta \leq 2\pi,$$

traversed counterclockwise.

Tasks

1. Begin with $A = \frac{1}{2} \oint_C (x \, dy - y \, dx)$.
2. Show that $x \, dy - y \, dx = r^2 \, d\theta$.
3. Evaluate the resulting trigonometric integral.

Question 6 – Solution

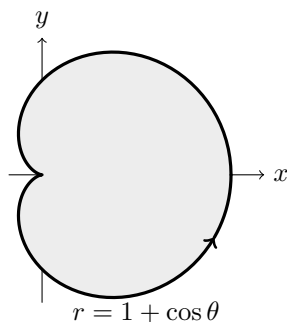
Strategy. Combine Green's area formula with polar boundary coordinates.

Step 1: Convert the differential With $x = r \cos \theta$ and $y = r \sin \theta$,

$$dx = (r' \cos \theta - r \sin \theta) d\theta, \quad dy = (r' \sin \theta + r \cos \theta) d\theta.$$

The rr' terms cancel, leaving

$$x dy - y dx = r^2(\cos^2 \theta + \sin^2 \theta) d\theta = r^2 d\theta.$$



Step 2: Evaluate the area

$$\begin{aligned} A &= \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 d\theta \\ &= \frac{1}{2} \int_0^{2\pi} (1 + 2 \cos \theta + \cos^2 \theta) d\theta \\ &= \frac{1}{2} (2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}. \end{aligned}$$

Verification The standard polar area formula is $\frac{1}{2} \int r^2 d\theta$, exactly what Green's Theorem produced. The nonnegative integrand confirms the positive area.

Question 7

Let C be the counterclockwise triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$. For

$$P(x, y) = -y^2, \quad Q(x, y) = x^2,$$

verify Green's Theorem by computing $\oint_C P dx + Q dy$ both directly and as a double integral.

Tasks

1. Evaluate all three boundary segments directly.
2. Evaluate $\iint_D (Q_x - P_y) dA$.
3. Compare the two results.

Question 7 – Solution

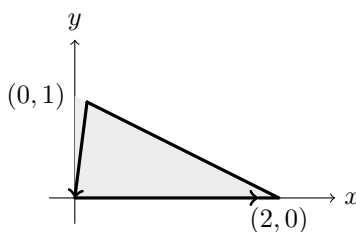
Strategy. Only the slanted edge contributes directly; then use $0 \leq y \leq 1 - x/2$ for the interior.

Step 1: Boundary calculation The horizontal and vertical coordinate-axis edges contribute zero. Parametrize the slanted edge from $(2, 0)$ to $(0, 1)$ by

$$\mathbf{r}(t) = \langle 2 - 2t, t \rangle, \quad 0 \leq t \leq 1.$$

Then $dx = -2dt$, $dy = dt$, and

$$\int_{\text{slant}} P dx + Q dy = \int_0^1 [2t^2 + 4(1 - t)^2] dt = \boxed{2}.$$



Step 2: Interior calculation Since $Q_x - P_y = 2x + 2y$,

$$\begin{aligned} \iint_D (2x + 2y) dA &= \int_0^2 \int_0^{1-x/2} (2x + 2y) dy dx \\ &= \int_0^2 \left(1 + x - \frac{3x^2}{4} \right) dx = \boxed{2}. \end{aligned}$$

Verification Both methods give 2. The slanted parametrization runs from $(2, 0)$ to $(0, 1)$, which is the required counterclockwise direction after traversing the bottom edge.

Question 8

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay^2, x^2 \rangle.$$

The counterclockwise circulation around the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is -2 . Determine a .

Tasks

1. Express the circulation as a double integral.
2. Solve for a .
3. Verify uniqueness and the stated circulation.

Question 8 – Solution

Strategy. Green's Theorem turns the circulation condition into a linear equation for a .

Step 1: Form the integrand With $P = ay^2$ and $Q = x^2$,

$$Q_x - P_y = 2x - 2ay.$$

Thus

$$\begin{aligned}\oint_C \mathbf{F}_a \cdot d\mathbf{r} &= \int_0^1 \int_0^1 (2x - 2ay) dy dx \\ &= \int_0^1 (2x - a) dx = 1 - a.\end{aligned}$$

Step 2: Impose the condition

$$1 - a = -2 \quad \implies \quad \boxed{a = 3}.$$

Step 3: Verify For $a = 3$,

$$\int_0^1 \int_0^1 (2x - 6y) dy dx = 1 - 3 = -2,$$

exactly the specified counterclockwise circulation.

Uniqueness check The circulation $1 - a$ has nonzero coefficient -1 on a , so the equation has exactly one solution.

Question 9

Let C be a positively oriented, simple, piecewise smooth closed curve enclosing a region D . Derive the three area formulas

$$A = \oint_C x \, dy, \quad A = - \oint_C y \, dx, \quad A = \frac{1}{2} \oint_C (x \, dy - y \, dx).$$

Tasks

1. Choose P, Q for each of the first two formulas.
2. Derive the symmetric formula from Green's Theorem or by averaging.
3. Explain the orientation requirement.

Question 9 – Solution

Strategy. Make the Green's Theorem integrand $Q_x - P_y$ equal to 1 in three different ways.

Step 1: Formula using $x \, dy$ Choose $P = 0$, $Q = x$. Then $Q_x - P_y = 1$, so

$$\text{Area}(D) = \iint_D 1 \, dA = \boxed{\oint_C x \, dy}.$$

Step 2: Formula using $-y \, dx$ Choose $P = -y$, $Q = 0$. Again $Q_x - P_y = 1$, giving

$$\text{Area}(D) = \boxed{-\oint_C y \, dx}.$$

Step 3: Symmetric formula Choose $P = -y/2$ and $Q = x/2$. Then

$$Q_x - P_y = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1,$$

and therefore

$$\boxed{\text{Area}(D) = \frac{1}{2} \oint_C (x \, dy - y \, dx)}.$$

This is also the average of the first two formulas.

Verification Green's Theorem uses positive, counterclockwise orientation. Clockwise traversal negates all three line integrals, so a minus sign would be needed to recover positive geometric area.

Question 10

Let D be the ellipse bounded counterclockwise by

$$x = 2 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq 2\pi.$$

Evaluate

$$\iint_D (3x^2 - 2y) \, dA$$

by converting it to a boundary integral.

Tasks

1. Choose P, Q so that $Q_x - P_y = 3x^2 - 2y$.
2. Parametrize the resulting boundary integral.
3. Evaluate using symmetry and a standard trigonometric integral.

Question 10 – Solution

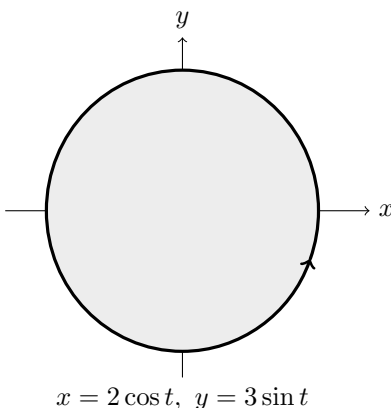
Strategy. Reverse Green's Theorem with $P = y^2$ and $Q = x^3$, then use the supplied ellipse parametrization.

Step 1: Choose a boundary field Set

$$P = y^2, \quad Q = x^3.$$

Then $Q_x - P_y = 3x^2 - 2y$, so

$$\iint_D (3x^2 - 2y) dA = \oint_C y^2 dx + x^3 dy.$$



Step 2: Substitute the boundary Since $dx = -2 \sin t dt$ and $dy = 3 \cos t dt$,

$$\oint_C y^2 dx + x^3 dy = \int_0^{2\pi} [-18 \sin^3 t + 24 \cos^4 t] dt.$$

The sine-cubed term integrates to zero over a full period, while

$$\int_0^{2\pi} \cos^4 t dt = \frac{3\pi}{4}.$$

Therefore

$$\boxed{\iint_D (3x^2 - 2y) dA = 24 \left(\frac{3\pi}{4} \right) = 18\pi}.$$

Verification The $-2y$ term is odd across the x -axis and contributes zero. Also $3x^2 \geq 0$, so the positive result has the required sign.