

Question 1

Determine whether

$$\mathbf{F}(x, y) = \langle 2xy + 3, x^2 + 4y \rangle$$

is conservative on $\mathbb{R}^2$. If it is, find a potential function f .

Tasks

1. Compare the relevant mixed partial derivatives.
2. Construct f and determine the one-variable correction term.
3. Differentiate the result to verify the field.

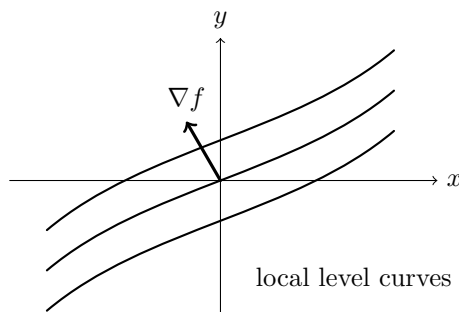
Question 1 – Solution

Strategy. Test $P_y = Q_x$ on the simply connected domain, then integrate one component and recover the missing function.

Step 1: Test the field With $P = 2xy + 3$ and $Q = x^2 + 4y$,

$$P_y = 2x, \quad Q_x = 2x.$$

The components have continuous first partial derivatives on all of $\mathbb{R}^2$, which is simply connected. Thus the equality guarantees that $\mathbf{F}$ is conservative.



Step 2: Construct a potential Integrating $f_x = P$ with respect to x gives

$$f = x^2y + 3x + g(y).$$

Then $f_y = x^2 + g'(y) = Q = x^2 + 4y$, so $g'(y) = 4y$ and $g(y) = 2y^2 + C$.

$$\boxed{f(x, y) = x^2y + 3x + 2y^2 + C}.$$

Verification Differentiation gives $\nabla f = \langle 2xy + 3, x^2 + 4y \rangle = \mathbf{F}$. The arbitrary constant has zero gradient.

Question 2

Consider

$$\mathbf{F}(x, y) = \langle y^2, 2xy + x \rangle$$

on $\mathbb{R}^2$.

Tasks

1. Test whether $\mathbf{F}$ is conservative.
2. Identify the exact obstruction.
3. Explain why no potential function can exist even locally near any point.

Question 2 – Solution

Strategy. A continuously differentiable gradient field must satisfy equality of the cross partials.

Step 1: Compute the cross partials Let $P = y^2$ and $Q = 2xy + x$. Then

$$P_y = 2y, \quad Q_x = 2y + 1.$$

Since $P_y \neq Q_x$ at every point, the necessary condition fails.

$\mathbf{F}$ is not conservative on $\mathbb{R}^2$.

Step 2: Identify the contradiction If $\mathbf{F} = \nabla f$ for a twice continuously differentiable f , then

$$f_x = P, \quad f_y = Q,$$

and hence $f_{xy} = P_y$ and $f_{yx} = Q_x$. Equality of mixed partials would require $2y = 2y + 1$, which is impossible.

Step 3: Local conclusion The mismatch is the constant 1 everywhere, not merely at an isolated point. Every open neighborhood contains points where the necessary equality fails, so no local potential exists on any such neighborhood.

Verification Integrating P gives $f = xy^2 + g(y)$, whose y -derivative is $2xy + g'(y)$. Matching Q would require $g'(y) = x$, impossible because g' cannot depend on x .

Question 3

For a real constant a , let

$$\mathbf{F}_a(x, y) = \langle ay + 2xy, 3x + x^2 \rangle.$$

Determine the value of a for which $\mathbf{F}_a$ is conservative on $\mathbb{R}^2$, and find a potential for that value.

Tasks

1. Use the mixed-partial condition to determine a .
2. Construct a potential function.
3. Verify both components and uniqueness of the parameter.

Question 3 – Solution

Strategy. Make the cross-partial identity hold for every point, not just at a selected point.

Step 1: Determine the parameter Write $P = ay + 2xy$ and $Q = 3x + x^2$. Then

$$P_y = a + 2x, \quad Q_x = 3 + 2x.$$

Equality for all x requires

$$\boxed{a = 3}.$$

No other constant can make the two affine expressions identical.

Step 2: Construct the potential For $a = 3$,

$$f_x = 3y + 2xy.$$

Integrating with respect to x gives

$$f = 3xy + x^2y + g(y).$$

Now $f_y = 3x + x^2 + g'(y)$ must equal $Q = 3x + x^2$, so $g'(y) = 0$.

$$\boxed{f(x, y) = 3xy + x^2y + C}.$$

Verification The gradient is $\langle 3y + 2xy, 3x + x^2 \rangle$, exactly $\mathbf{F}_3$. The coefficient comparison also proves the value $a = 3$ is unique.

Question 4

On the punctured plane $D = \mathbb{R}^2 \setminus \{(0, 0)\}$, consider

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle.$$

Investigate whether $\mathbf{F}$ is conservative on D .

Tasks

1. Verify that $P_y = Q_x$ on D .
2. Evaluate the line integral around the counterclockwise unit circle.
3. Reconcile the two results using the geometry of the domain.

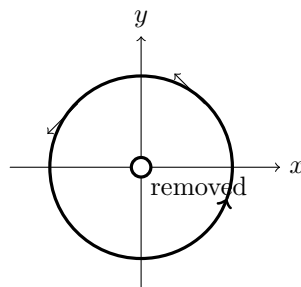
Question 4 – Solution

Strategy. The cross-partial test needs a suitable domain; use a closed-loop integral to detect the hole.

Step 1: Cross partials For $(x, y) \neq (0, 0)$,

$$P_y = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \quad Q_x = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Thus the local equality holds throughout D .



$\mathbb{R}^2 \setminus \{(0, 0)\}$ has a hole

Step 2: Test a closed circle Set $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t),$$

so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 1 \, dt = \boxed{2\pi}.$$

A conservative field must have zero integral around every closed curve. Therefore

$$\boxed{\mathbf{F} \text{ is not conservative on } D}.$$

Step 3: Reconcile The punctured plane is not simply connected: the unit circle encloses the missing origin and cannot contract to a point within D . Consequently $P_y = Q_x$ is not sufficient here.

Question 5

Restrict the field

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle$$

to the right half-plane $H = \{(x, y) : x > 0\}$.

Tasks

1. Find a potential function on H .
2. Verify its gradient.
3. Evaluate the line integral from $A = (1, 1)$ to $B = (\sqrt{3}, 1)$ along any path in H .

Question 5 – Solution

Strategy. The right half-plane permits a single-valued angle function with no branch crossing.

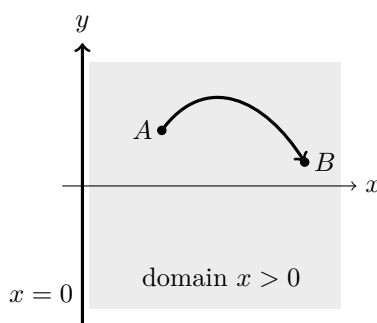
Step 1: Choose the angle potential On $x > 0$, define

$$f(x, y) = \arctan\left(\frac{y}{x}\right).$$

Using the chain rule,

$$f_x = \frac{-y}{x^2 + y^2}, \quad f_y = \frac{x}{x^2 + y^2}.$$

Hence $\nabla f = \mathbf{F}$ on H .



Step 2: Evaluate by endpoints The endpoint angles in the branch $(-\pi/2, \pi/2)$ are

$$f(A) = \arctan(1) = \frac{\pi}{4}, \quad f(B) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

Therefore

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = -\frac{\pi}{12}}.$$

Verification Both endpoints and the entire allowed path lie in $x > 0$, where the selected arctangent branch is continuous. The negative sign matches the decrease in polar angle from $\pi/4$ to $\pi/6$.

Question 6

Determine whether the three-dimensional field

$$\mathbf{F}(x, y, z) = \langle 2xy + z, x^2 + 2y, x + 2z \rangle$$

is conservative on $\mathbb{R}^3$. If so, find a potential and use it to evaluate the integral from $(0, 0, 0)$ to $(1, 2, 3)$.

Tasks

1. Check the three pairwise cross-partial relations.
2. Construct and verify a potential.
3. Evaluate the requested line integral by endpoints.

Question 6 – Solution

Strategy. Verify compatibility of all three components, then integrate successively while retaining correction functions.

Step 1: Compatibility Let $\mathbf{F} = \langle P, Q, R \rangle$. Then

$$P_y = 2x = Q_x, \quad P_z = 1 = R_x, \quad Q_z = 0 = R_y.$$

On the simply connected domain $\mathbb{R}^3$, these identities show the field is conservative.

Step 2: Construct f Integrating $f_x = P$ gives

$$f = x^2y + xz + g(y, z).$$

From $f_y = x^2 + g_y = Q$, we get $g_y = 2y$, so $g = y^2 + h(z)$. Finally,

$$f_z = x + h'(z) = R = x + 2z,$$

so $h = z^2 + C$. Thus

$$\boxed{f = x^2y + xz + y^2 + z^2 + C}.$$

Step 3: Endpoint evaluation

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 3) - f(0, 0, 0) = 2 + 3 + 4 + 9 = \boxed{18}.$$

Verification Differentiating the boxed potential returns all three original components; the additive constant cancels from the endpoint difference.

Question 7

Suppose f and g are potential functions for the same continuous vector field $\mathbf{F}$ on a connected open region D .

Tasks

1. Prove that $f - g$ is constant on D .
2. Explain why connectedness is needed.
3. State the resulting uniqueness principle for potential functions.

Question 7 – Solution

Strategy. Subtract the two gradient equations and use paths inside the connected open region.

Step 1: Subtract gradients Since $\nabla f = \mathbf{F} = \nabla g$,

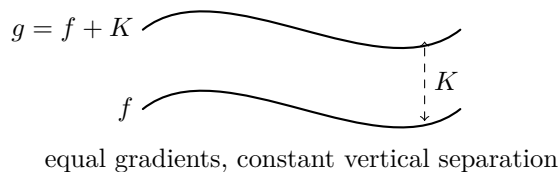
$$\nabla(f - g) = \mathbf{0}.$$

Let $A, B \in D$. An open connected region in $\mathbb{R}^n$ is path connected, so choose a piecewise smooth curve C in D from A to B . Then

$$(f - g)(B) - (f - g)(A) = \int_C \nabla(f - g) \cdot d\mathbf{r} = 0.$$

Thus $(f - g)(B) = (f - g)(A)$ for every pair, and

$$\boxed{f - g = K \text{ on } D}.$$



Step 2: Why connectedness matters On separate components, the same zero gradient permits a different constant on each component because no path in D joins them.

Step 3: Uniqueness principle A potential on a connected region is unique up to one additive constant. Adding a constant changes function values but leaves every partial derivative unchanged.

Verification If $g = f + K$, then $\nabla g = \nabla f + \nabla K = \nabla f$, confirming that every additive constant really does produce another potential.

Question 8

Let $\mathbf{F}$ be continuous on a region where the indicated curves lie. Prove that the following statements are equivalent:

1. line integrals of $\mathbf{F}$ are path independent;
2. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C .

Tasks

1. Prove path independence implies the closed-loop condition.
2. Prove the converse using two paths with common endpoints.
3. Track orientation carefully in the concatenated curve.

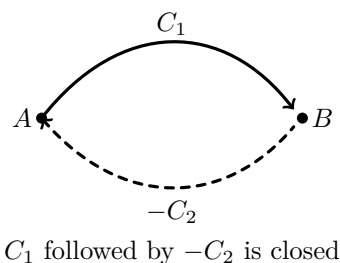
Question 8 – Solution

Strategy. A closed curve can be compared with a constant path, while two endpoint-sharing paths form a closed curve when one is reversed.

Step 1: Path independence $\Rightarrow$ zero loops Let a closed curve start and end at A . Compare it with the constant path at A , whose line integral is zero. Path independence gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Step 2: Zero loops $\Rightarrow$ path independence Let C_1 and C_2 both run from A to B . Follow C_1 from A to B , then follow $-C_2$ back to A . This concatenation is closed.



By the assumed closed-loop condition,

$$0 = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Therefore

$$\boxed{\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}},$$

which is path independence.

Verification The reversal is essential: following C_2 in its original direction would begin at A , not at the terminal point B of C_1 , and would not close the loop.

Question 9

A field on $\mathbb{R}^2$ has the form

$$\mathbf{F}(x, y) = \langle 2xy + y, Q(x, y) \rangle.$$

It is conservative and satisfies $Q(0, y) = e^y$.

Tasks

1. Determine the most general Q allowed by the mixed-partial condition.
2. Use the boundary data to determine Q uniquely.
3. Find and verify a potential function.

Question 9 – Solution

Strategy. Integrate the compatibility equation for the unknown component, then use the supplied trace data to fix its arbitrary function.

Step 1: Compatibility equation With $P = 2xy + y$,

$$P_y = 2x + 1.$$

Conservativeness requires $Q_x = P_y$, so integrating with respect to x gives

$$Q(x, y) = x^2 + x + h(y).$$

Step 2: Apply the boundary data At $x = 0$,

$$Q(0, y) = h(y) = e^y.$$

Thus

$$\boxed{Q(x, y) = x^2 + x + e^y}.$$

Step 3: Find a potential Integrating $f_x = P$ yields

$$f = x^2y + xy + g(y).$$

Then

$$f_y = x^2 + x + g'(y) = Q$$

requires $g'(y) = e^y$, so

$$\boxed{f(x, y) = x^2y + xy + e^y + C}.$$

Verification Its gradient is $\langle 2xy + y, x^2 + x + e^y \rangle$, and setting $x = 0$ in the second component gives exactly e^y .

Question 10

Classify the linear vector fields

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle,$$

where a, b, c, d are real constants.

Tasks

1. Find the necessary and sufficient condition for $\mathbf{F}$ to be conservative on $\mathbb{R}^2$.
2. Construct the general potential when the condition holds.
3. Under that condition, compute the work from $(0, 0)$ to (p, q) .

Question 10 – Solution

Strategy. Compare the constant cross partials, then integrate symbolically without assigning numerical coefficients.

Step 1: Classify Let $P = ax + by$ and $Q = cx + dy$. Then

$$P_y = b, \quad Q_x = c.$$

Because $\mathbb{R}^2$ is simply connected, equality is both necessary and sufficient. Hence

$\mathbf{F}$ is conservative exactly when $b = c$.

Step 2: Construct the potential Assume $b = c$. Integrating $f_x = ax + by$ gives

$$f = \frac{a}{2}x^2 + bxy + g(y).$$

Since $f_y = bx + g'(y) = bx + dy$, we have $g'(y) = dy$, so

$f(x, y) = \frac{a}{2}x^2 + bxy + \frac{d}{2}y^2 + C.$

Step 3: Compute the work The Fundamental Theorem for Line Integrals gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(p, q) - f(0, 0) = \left[\frac{a}{2}p^2 + bpq + \frac{d}{2}q^2 \right].$$

Verification Differentiating the potential gives $\langle ax + by, bx + dy \rangle$, which equals the original field precisely because $c = b$. The additive constant cancels from the work.