

Question 1

Let

$$f(x, y, z) = x^2y + e^z,$$

and let C be any smooth curve from $A = (1, 0, 0)$ to $B = (2, 1, \ln 2)$. Evaluate

$$\int_C \nabla f \cdot d\mathbf{r}.$$

Tasks

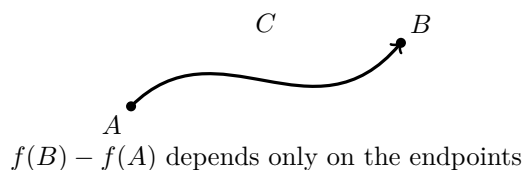
1. State the Fundamental Theorem for Line Integrals.
2. Evaluate the integral without parametrizing C .
3. Explain why the unspecified shape of C is irrelevant.

Question 1 – Solution

Strategy. Since the field is explicitly ∇f , apply the endpoint formula $\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$.

Step 1: Evaluate the endpoint values

$$f(B) = 2^2(1) + e^{\ln 2} = 4 + 2 = 6, \quad f(A) = 1^2(0) + e^0 = 1.$$



Step 2: Apply the theorem

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 6 - 1 = \boxed{5}.$$

Step 3: Explain path independence Along any smooth parametrization $\mathbf{r}(t)$, $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ is the derivative of $f(\mathbf{r}(t))$. Its integral therefore records only the net change in f .

Verification The logarithmic coordinate was chosen so that $e^{\ln 2} = 2$; using $\ln 2$ itself in $f(B)$ would be an endpoint-substitution error.

Question 2

Let $f(x, y) = x^2 + y^2$. A smooth curve C travels from $A = (3, 4)$ to $B = (0, 5)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$.
2. Interpret the result using level curves of f .
3. Decide whether a different smooth path from A to B could change the value.

Question 2 – Solution

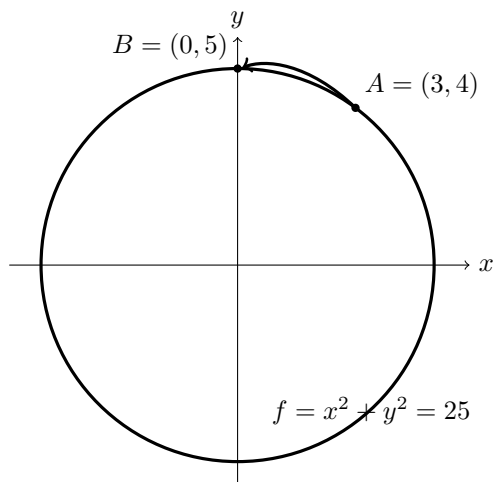
Strategy. Compare the potential values at the endpoints before doing any curve calculation.

Step 1: Endpoint values

$$f(A) = 3^2 + 4^2 = 25, \quad f(B) = 0^2 + 5^2 = 25.$$

Hence

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 0.}$$



Step 2: Interpret Both endpoints lie on the level circle $x^2 + y^2 = 25$. The curve between them need not stay on that circle: f may rise and fall along the way, but its total change is still zero.

Step 3: Compare paths Any other smooth path with the same orientation from A to B gives the same endpoint difference, so it cannot change the integral.

Verification The result is not claiming that ∇f vanishes. Indeed $\nabla f = \langle 2x, 2y \rangle$ is nonzero at both endpoints; only the net change in f is zero.

Question 3

Let

$$f(x, y, z) = xy + z^2,$$

and let C be any piecewise smooth closed curve contained in the domain of f . Evaluate

$$\oint_C \nabla f \cdot d\mathbf{r}.$$

Tasks

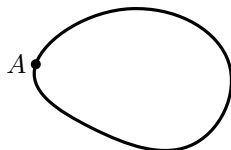
1. Apply the endpoint form of the theorem to the closed curve.
2. Explain the role of the curve's initial and terminal points.
3. State whether self-intersections affect the result.

Question 3 – Solution

Strategy. Regard a closed curve as an oriented path whose terminal point equals its initial point.

Step 1: Name the common endpoint Let A be the point at which the parametrization starts and ends. The theorem gives

$$\oint_C \nabla f \cdot d\mathbf{r} = f(A) - f(A) = \boxed{0}.$$



initial point = terminal point

Step 2: Interpret The integral measures the accumulated change of f during one traversal. Returning to the starting point returns to the same value of f , so all increases and decreases cancel.

Step 3: Self-intersections Self-intersections do not alter the endpoint calculation. Each piece contributes the change in f across that piece, and the intermediate values telescope when the pieces are added.

Verification Reversing the closed curve would negate its integral, but the negative of zero is still zero, consistent with the answer.

Question 4

Let $f(x, y) = xe^y$. A curve C is oriented from $A = (0, 0)$ to $B = (2, \ln 3)$, and $-C$ denotes the same curve with the reverse orientation.

Tasks

1. Compute $\int_C \nabla f \cdot d\mathbf{r}$.
2. Compute $\int_{-C} \nabla f \cdot d\mathbf{r}$.
3. Explain the sign change using both endpoints and orientation reversal.

Question 4 – Solution

Strategy. Evaluate the same endpoint difference in the two opposite orders.

Step 1: Forward orientation

$$f(A) = 0e^0 = 0, \quad f(B) = 2e^{\ln 3} = 6.$$

Thus

$$\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = \boxed{6}.$$

Step 2: Reverse orientation The initial point of $-C$ is B and its terminal point is A , so

$$\int_{-C} \nabla f \cdot d\mathbf{r} = f(A) - f(B) = \boxed{-6}.$$

Step 3: Connect the rules Swapping the endpoints reverses the subtraction. This is exactly the general orientation rule

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Verification Adding the forward and reverse traversals makes a closed trip, and $6 + (-6) = 0$, as the theorem requires for a gradient field.

Question 5

Let $f(x, y) = x^2y$ and let C be the quarter unit circle

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ in two ways.

Tasks

1. Use the Fundamental Theorem for Line Integrals.
2. Evaluate directly from $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$.
3. Identify the chain-rule antiderivative in the direct calculation.

Question 5 – Solution

Strategy. First use the endpoints, then verify that the parameter integrand is the derivative of $f(\mathbf{r}(t))$.

Step 1: Endpoint method The curve begins at $(1, 0)$ and ends at $(0, 1)$. Both endpoint values are zero:

$$f(1, 0) = 0, \quad f(0, 1) = 0.$$

Therefore the theorem gives $\boxed{0}$.

Step 2: Direct method Since $\nabla f = \langle 2xy, x^2 \rangle$,

$$\begin{aligned}\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle 2 \cos t \sin t, \cos^2 t \rangle \cdot \langle -\sin t, \cos t \rangle \\ &= -2 \cos t \sin^2 t + \cos^3 t.\end{aligned}$$

This is

$$\frac{d}{dt}(\cos^2 t \sin t) = \frac{d}{dt}f(\mathbf{r}(t)).$$

Hence

$$\int_0^{\pi/2} \frac{d}{dt}(\cos^2 t \sin t) dt = [\cos^2 t \sin t]_0^{\pi/2} = \boxed{0}.$$

Verification The two methods agree, and the displayed antiderivative explicitly exhibits the chain rule behind the theorem.

Question 6

Let $f(x, y) = x^2 + y^2$. A curve begins at $A = (1, 2)$ and ends at a first-quadrant point B on the line $y = x + 1$. Suppose

$$\int_C \nabla f \cdot d\mathbf{r} = 20.$$

Determine B .

Tasks

1. Convert the work condition into a level-curve condition on B .
2. Solve the line-circle intersection equation.
3. Use the quadrant restriction to select and verify the endpoint.

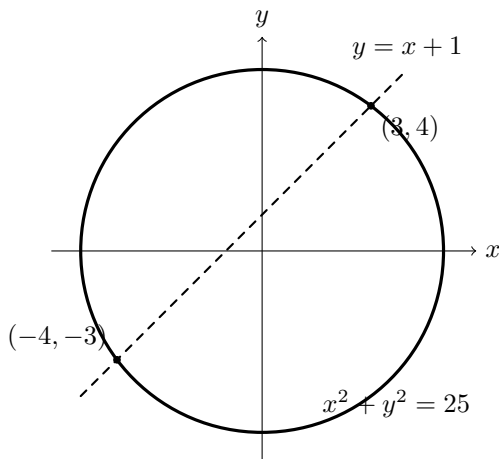
Question 6 – Solution

Strategy. Use the theorem in reverse: the known integral determines the terminal potential value.

Step 1: Determine the level Since $f(A) = 1^2 + 2^2 = 5$,

$$20 = f(B) - 5, \quad f(B) = 25.$$

Thus B lies on $x^2 + y^2 = 25$ as well as $y = x + 1$.



Step 2: Solve the intersection Substitute $y = x + 1$:

$$x^2 + (x + 1)^2 = 25 \implies 2x^2 + 2x - 24 = 0 \implies (x - 3)(x + 4) = 0.$$

The candidates are $(3, 4)$ and $(-4, -3)$.

Step 3: Apply the restriction Only $(3, 4)$ lies in the first quadrant, so

$$\boxed{B = (3, 4)}.$$

Verification The line condition gives $4 = 3 + 1$, and $f(B) - f(A) = 25 - 5 = 20$, exactly the specified integral.

Question 7

Let

$$f(x, y) = x^3 - 3xy + y^2.$$

A piecewise smooth path C travels from $A = (0, 0)$ to $(1, 0)$, then to $(1, 1)$, and finally to $B = (2, 1)$.

Tasks

1. Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ using the theorem.
2. Show how the endpoint changes telescope across the three pieces.
3. Explain why the corners do not invalidate the theorem.

Question 7 – Solution

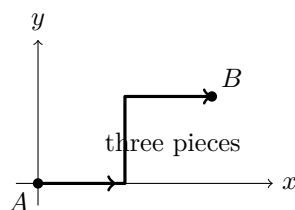
Strategy. Apply the theorem to the whole piecewise smooth path, then expose the same cancellation piece by piece.

Step 1: Whole-path calculation

$$f(A) = 0, \quad f(B) = 2^3 - 3(2)(1) + 1^2 = 8 - 6 + 1 = 3.$$

Therefore

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = 3}.$$



Step 2: Telescope Let $P = (1, 0)$ and $Q = (1, 1)$. Then

$$\begin{aligned} \int_C \nabla f \cdot d\mathbf{r} &= [f(P) - f(A)] + [f(Q) - f(P)] + [f(B) - f(Q)] \\ &= f(B) - f(A) = 3. \end{aligned}$$

Every intermediate potential value appears once positively and once negatively.

Step 3: Corners Smoothness is required on each segment, not at the finitely many joining corners. Adding the valid integral on each piece yields the integral on the piecewise smooth path.

Verification Direct values $f(P) = 1$ and $f(Q) = -1$ give contributions 1, -2 , 4, whose sum is 3.

Question 8

Let

$$f(x, y, z) = xyz + \frac{1}{2}(x^2 + y^2 + z^2).$$

Evaluate $\int_C \nabla f \cdot d\mathbf{r}$ for any smooth space curve from $A = (1, -1, 0)$ to $B = (2, 1, 3)$.

Tasks

1. Compute $f(A)$ and $f(B)$ carefully.
2. Apply the Fundamental Theorem for Line Integrals.
3. Verify that the associated vector field is three-dimensional by computing ∇f .

Question 8 – Solution

Strategy. The spatial geometry of C is unnecessary because the integrand is the gradient of the supplied scalar function.

Step 1: Initial value

$$f(1, -1, 0) = (1)(-1)(0) + \frac{1}{2}(1 + 1 + 0) = 1.$$

Step 2: Terminal value

$$f(2, 1, 3) = (2)(1)(3) + \frac{1}{2}(4 + 1 + 9) = 6 + 7 = 13.$$

Thus

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A) = 13 - 1 = 12}.$$

Step 3: Display the field Differentiating componentwise gives

$$\nabla f = \langle yz + x, xz + y, xy + z \rangle.$$

This confirms that the line integral uses all three coordinate components, even though no parametrization is needed.

Verification The quadratic contribution changes from 1 to 7, a gain of 6, and the product contribution changes from 0 to 6, another gain of 6. The total change is 12.

Question 9

A force is derived from the potential energy

$$U(x, y) = x^2 + 2y^2$$

by $\mathbf{F} = -\nabla U$. A particle moves along any smooth curve from $A = (2, 1)$ to $B = (0, 3)$.

Tasks

1. Compute the work $W = \int_C \mathbf{F} \cdot d\mathbf{r}$.
2. Relate the work to the change in potential energy.
3. Explain the sign of the answer.

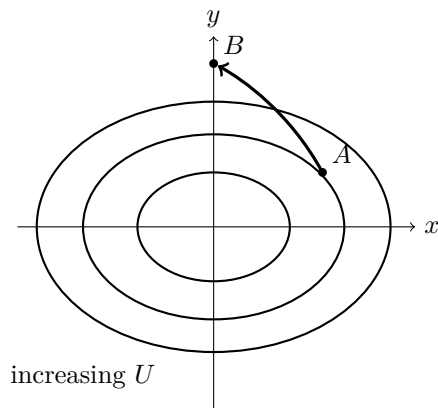
Question 9 – Solution

Strategy. Apply the theorem to $-U$: work by $-\nabla U$ is the negative of the change in U .

Step 1: Potential-energy change

$$U(A) = 2^2 + 2(1^2) = 6, \quad U(B) = 0^2 + 2(3^2) = 18.$$

Therefore $\Delta U = U(B) - U(A) = 12$.



Step 2: Work Since $\mathbf{F} = \nabla(-U)$,

$$\begin{aligned} W &= (-U)(B) - (-U)(A) \\ &= U(A) - U(B) = 6 - 18 = \boxed{-12}. \end{aligned}$$

Equivalently, $W = -\Delta U = -12$.

Step 3: Interpret the sign The particle ends at a point with greater potential energy. The force derived from $-\nabla U$ therefore does negative net work during the motion.

Verification The identity $W + \Delta U = -12 + 12 = 0$ checks the sign. Reversing the trip would produce work $+12$.

Question 10

Let f have continuous first partial derivatives on an open region containing a smooth curve $\mathbf{r}(t)$, $a \leq t \leq b$.
Prove

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Tasks

1. Rewrite the line integral using the parametrization.
2. Apply the multivariable chain rule.
3. Finish with the one-variable Fundamental Theorem of Calculus and identify the hypotheses used.

Question 10 – Solution

Strategy. Show that the vector-field integrand is exactly the ordinary derivative of the composite function $f \circ \mathbf{r}$.

Step 1: Parameter form Write $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. By definition,

$$\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

Expanding the dot product gives

$$\int_a^b (f_x x'(t) + f_y y'(t) + f_z z'(t)) dt,$$

where the partial derivatives are evaluated at $\mathbf{r}(t)$.

Step 2: Apply the chain rule The multivariable chain rule states

$$\frac{d}{dt} f(\mathbf{r}(t)) = f_x x'(t) + f_y y'(t) + f_z z'(t).$$

Thus the line integral is $\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt$.

Step 3: Apply the one-variable theorem

$$\int_a^b \frac{d}{dt} f(\mathbf{r}(t)) dt = \boxed{f(\mathbf{r}(b)) - f(\mathbf{r}(a))}.$$

The smoothness of $\mathbf{r}$ and continuity of the first partial derivatives make the chain-rule derivative continuous and integrable. The same proof applies piece by piece to a piecewise smooth curve, with intermediate endpoint values canceling.

Verification The right side reverses sign when a and b are interchanged, matching orientation reversal of the line integral.