

Question 1

Let

$$\mathbf{F}(x, y) = \langle y, x^2 \rangle, \quad \mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

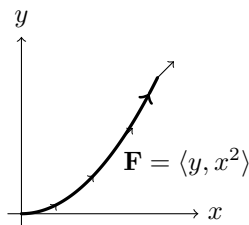
1. Evaluate $\mathbf{F}$ along the curve and compute $\mathbf{r}'(t)$.
2. Form the dot product and evaluate the integral.
3. State how the sign of the integrand agrees with the geometry.

Question 1 – Solution

Strategy. Use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$.

Step 1: Restrict the field to the curve Since $x = t$ and $y = t^2$,

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^2, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t \rangle.$$



Step 2: Form the scalar integrand

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^2 + 2t^3.$$

Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^3) dt = \left[\frac{t^3}{3} + \frac{t^4}{2} \right]_0^1 = \boxed{\frac{5}{6}}.$$

Verification For $0 \leq t \leq 1$, both components of the field and both components of the velocity are nonnegative. Their dot product must be nonnegative, consistent with the positive answer.

Question 2

A particle follows the helix

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq 2\pi,$$

through the constant force field $\mathbf{F} = \langle 2, -1, 3 \rangle$.

Tasks

1. Compute the work done by the force.
2. Identify the contributions from the horizontal and vertical motion.
3. Check the result using the net displacement.

Question 2 – Solution

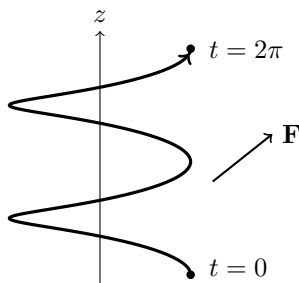
Strategy. Dot the constant force with the velocity, then use the displacement as an independent check.

Step 1: Velocity and dot product

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

so

$$\mathbf{F} \cdot \mathbf{r}'(t) = -2 \sin t - \cos t + 3.$$

**Step 2: Work**

$$\begin{aligned} W &= \int_0^{2\pi} (-2 \sin t - \cos t + 3) dt \\ &= 0 + 0 + 6\pi = \boxed{6\pi}. \end{aligned}$$

The horizontal coordinates complete a full turn, so their signed contributions cancel. The vertical rise is 2π , producing work $3(2\pi)$.

Verification The net displacement is $\mathbf{r}(2\pi) - \mathbf{r}(0) = \langle 0, 0, 2\pi \rangle$. For a constant force, its dot product with this displacement is $\langle 2, -1, 3 \rangle \cdot \langle 0, 0, 2\pi \rangle = 6\pi$.

Question 3

Let C travel from $A = (0, 0)$ to $B = (2, 0)$ and then from B to $C = (2, 1)$. For

$$\mathbf{F}(x, y) = \langle -y, x \rangle,$$

evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

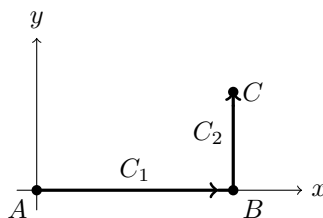
1. Parametrize each segment with the stated orientation.
2. Compute the contribution from each segment.
3. Explain geometrically why the first contribution vanishes.

Question 3 – Solution

Strategy. Split the piecewise path and add the two oriented line integrals.

Step 1: Horizontal segment Use $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq 2$. Then

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle 0, t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 0 \rangle, \quad \mathbf{F} \cdot \mathbf{r}'_1 = 0.$$



The field is vertical along this segment while the motion is horizontal, so the vectors are perpendicular.

Step 2: Vertical segment Use $\mathbf{r}_2(s) = \langle 2, s \rangle$, $0 \leq s \leq 1$. Thus

$$\mathbf{F}(\mathbf{r}_2(s)) = \langle -s, 2 \rangle, \quad \mathbf{r}'_2(s) = \langle 0, 1 \rangle,$$

and

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2 \, ds = 2.$$

Step 3: Add Hence

$$\boxed{\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 2 = 2}.$$

Verification Only the upward component on the vertical segment contributes, and that component is the constant 2 over a distance of 1.

Question 4

Let C be the unit circle and let

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ when C is traversed (a) counterclockwise and (b) clockwise.

Tasks

1. Parametrize the counterclockwise circle and evaluate the integral.
2. Use orientation reversal to obtain the clockwise value.
3. Interpret the signs using the direction of the field.

Question 4 – Solution

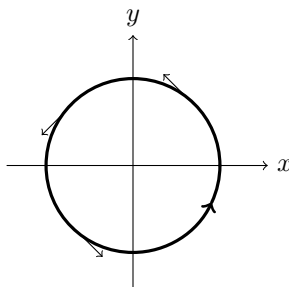
Strategy. On the unit circle this field is the counterclockwise unit tangent, making the dot product especially simple.

Step 1: Counterclockwise traversal Take

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq 2\pi.$$

Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle -\sin t, \cos t \rangle = \mathbf{r}'(t).$$



field and traversal are counterclockwise

Consequently,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{r}'(t) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} 1 dt = \boxed{2\pi}.$$

Step 2: Reverse the orientation Reversing a vector-field line integral changes its sign, so the clockwise value is

$$\boxed{-2\pi}.$$

Verification The field assists counterclockwise motion at every point and opposes clockwise motion at every point, so the respective signs must be positive and negative.

Question 5

Let

$$\mathbf{F}(x, y, z) = \langle z, x, y \rangle, \quad \mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad 0 \leq t \leq 1.$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Tasks

1. Evaluate the field along the space curve.
2. Compute and integrate the dot product.
3. Verify the calculation by expanding $z \, dx + x \, dy + y \, dz$.

Question 5 – Solution

Strategy. Substitute all three coordinates before taking the dot product with the three-dimensional velocity.

Step 1: Field and velocity

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^3, t, t^2 \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle.$$

Step 2: Dot and integrate

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= t^3 + 2t^2 + 3t^4, \\ \int_C \mathbf{F} \cdot d\mathbf{r} &= \left[\frac{t^4}{4} + \frac{2t^3}{3} + \frac{3t^5}{5} \right]_0^1 \\ &= \frac{1}{4} + \frac{2}{3} + \frac{3}{5} = \boxed{\frac{91}{60}}. \end{aligned}$$

Step 3: Differential-form check Along the curve,

$$z \, dx = t^3 \, dt, \quad x \, dy = t(2t \, dt) = 2t^2 \, dt, \quad y \, dz = t^2(3t^2 \, dt) = 3t^4 \, dt.$$

These are exactly the three terms above.

Verification Every term is nonnegative on $[0, 1]$, and the common-denominator sum $(15 + 40 + 36)/60 = 91/60$ confirms the arithmetic.

Question 6

For a constant a , let

$$\mathbf{F}_a(x, y) = \langle ay, x \rangle.$$

Along the parabola $\mathbf{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$, the work done is 2. Determine a .

Tasks

1. Express the work in terms of a .
2. Solve for a .
3. Substitute the result and verify the specified work.

Question 6 – Solution

Strategy. Restrict the one-parameter family of fields to the given curve; the work condition becomes a linear equation in a .

Step 1: Build the integrand

$$\mathbf{F}_a(\mathbf{r}(t)) = \langle at^2, t \rangle, \quad \mathbf{r}'(t) = \langle 1, 2t \rangle.$$

Thus

$$\mathbf{F}_a(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = at^2 + 2t^2 = (a + 2)t^2.$$

Step 2: Impose the work condition

$$2 = \int_0^1 (a + 2)t^2 dt = \frac{a + 2}{3}.$$

Therefore $a + 2 = 6$, and

$$\boxed{a = 4}.$$

Step 3: Verify With $a = 4$,

$$\int_0^1 (4 + 2)t^2 dt = 6 \left[\frac{t^3}{3} \right]_0^1 = 2,$$

which is the required work.

Uniqueness check The coefficient of a in the work is $\int_0^1 t^2 dt = 1/3 \neq 0$, so no second value of a can satisfy the condition.

Question 7

The field $\mathbf{F}(x, y) = \langle -y, x \rangle$ acts between $A = (0, 0)$ and $B = (1, 1)$. Compare the work along

$$C_1 : \mathbf{r}(t) = \langle t, t \rangle, \quad 0 \leq t \leq 1,$$

and the broken path C_2 that goes from $(0, 0)$ to $(1, 0)$ to $(1, 1)$.

Tasks

1. Compute the work along C_1 .
2. Compute the work along both pieces of C_2 .
3. Explain what the comparison demonstrates.

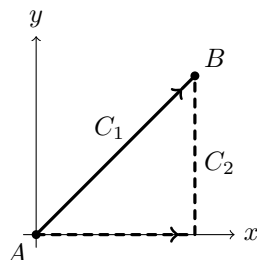
Question 7 – Solution

Strategy. Evaluate the same vector-field line integral on two explicitly parametrized paths with common endpoints.

Step 1: Diagonal path On C_1 ,

$$\mathbf{F}(\mathbf{r}(t)) = \langle -t, t \rangle, \quad \mathbf{r}'(t) = \langle 1, 1 \rangle.$$

The dot product is $-t + t = 0$, so $W_1 = \boxed{0}$.



Step 2: Broken path On the horizontal piece $\langle t, 0 \rangle$, the dot product $\langle 0, t \rangle \cdot \langle 1, 0 \rangle$ is zero. On the vertical piece $\langle 1, s \rangle$,

$$\langle -s, 1 \rangle \cdot \langle 0, 1 \rangle = 1, \quad 0 \leq s \leq 1.$$

Thus

$$W_2 = 0 + \int_0^1 1 \, ds = \boxed{1}.$$

Step 3: Compare Since $W_1 \neq W_2$ despite identical endpoints, the work for this field depends on the path taken.

Verification Along the diagonal the field is perpendicular to the motion. Along the final vertical piece of C_2 , its upward component is 1, explaining the extra unit of work.

Question 8

Let C be the counterclockwise boundary of the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$. For

$$\mathbf{F}(x, y) = \langle y^2, x \rangle,$$

compute the circulation $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly from the four edges.

Tasks

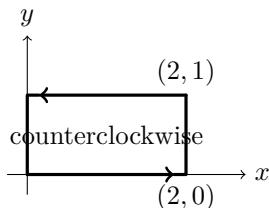
1. Compute each oriented edge contribution.
2. Add the four contributions.
3. Check which signs are forced by the traversal direction.

Question 8 – Solution

Strategy. Write $\mathbf{F} \cdot d\mathbf{r} = y^2 dx + x dy$ and exploit the constant coordinate on each edge.

Step 1: Bottom and right Along the bottom, $y = 0$ and $dy = 0$, so the contribution is 0. Along the right edge, $x = 2$ and $y : 0 \rightarrow 1$, giving

$$\int_0^1 2 dy = 2.$$



Step 2: Top and left Along the top, $y = 1$ and counterclockwise travel has $x : 2 \rightarrow 0$:

$$\int_2^0 1 dx = -2.$$

On the left, $x = 0$, and $dx = 0$, so the contribution is 0.

Step 3: Sum

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 2 - 2 + 0 = 0.$$

Verification The right-edge contribution is positive because both the field's vertical component and dy are positive. The top contribution is negative because $y^2 > 0$ while $dx < 0$; their equal magnitudes cancel.

Question 9

Let C be the line segment from $(0,0)$ to $(1,2)$ in the field

$$\mathbf{F}(x,y) = \langle x-y, x+y \rangle.$$

Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ using both

$$\mathbf{r}_1(t) = \langle t, 2t \rangle, \ 0 \leq t \leq 1, \quad \text{and} \quad \mathbf{r}_2(u) = \langle u^2, 2u^2 \rangle, \ 0 \leq u \leq 1.$$

Tasks

1. Evaluate the integral with each parametrization.
2. Explain why the zero velocity of $\mathbf{r}_2$ at $u = 0$ causes no problem.
3. State the invariance illustrated by the equal answers.

Question 9 – Solution

Strategy. Carry the parameter derivative through both computations; the slower start must be included in dr .

Step 1: First parametrization Here

$$\mathbf{F}(\mathbf{r}_1(t)) = \langle -t, 3t \rangle, \quad \mathbf{r}'_1(t) = \langle 1, 2 \rangle.$$

Thus

$$\int_0^1 (-t + 6t) dt = \int_0^1 5t dt = \boxed{\frac{5}{2}}.$$

Step 2: Nonuniform parametrization Now

$$\mathbf{F}(\mathbf{r}_2(u)) = \langle -u^2, 3u^2 \rangle, \quad \mathbf{r}'_2(u) = \langle 2u, 4u \rangle.$$

Therefore

$$\int_0^1 (-2u^3 + 12u^3) du = \int_0^1 10u^3 du = \boxed{\frac{5}{2}}.$$

Step 3: Interpret Although $\mathbf{r}'_2(0) = \mathbf{0}$, the curve pauses only at one endpoint and then traces the segment once with the same orientation. The derivative factor compensates for its varying speed.

Verification Both maps begin at $(0, 0)$, end at $(1, 2)$, and move monotonically along the same segment, so parameter independence requires the matching values obtained above.

Question 10

On the counterclockwise circle $C : x^2 + y^2 = 9$, consider

$$\mathbf{F}(x, y) = \langle x - 2y, 2x + y \rangle.$$

Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by separating $\mathbf{F}$ into radial and tangential parts.

Tasks

1. Decompose the field into radial and tangential vectors.
2. Determine the work contributed by each part.
3. Verify the answer by direct parametrization.

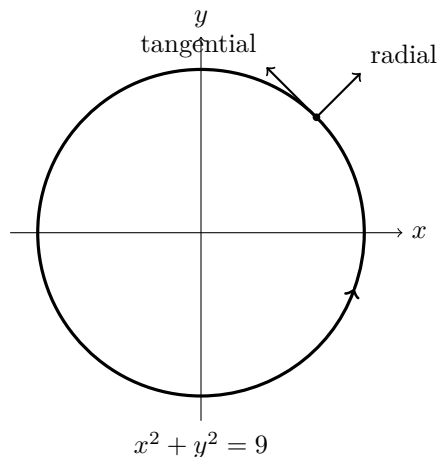
Question 10 – Solution

Strategy. Write the field as $\langle x, y \rangle + 2\langle -y, x \rangle$; on a circle the first part is normal and the second is tangent.

Step 1: Decompose

$$\mathbf{F} = \underbrace{\langle x, y \rangle}_{\text{radial}} + \underbrace{2\langle -y, x \rangle}_{\text{counterclockwise tangential}}.$$

The radial part is perpendicular to the circle's tangent, so it contributes zero work.



On the radius-3 circle, $\langle -y, x \rangle$ has magnitude 3 in the positive tangent direction. Hence the doubled part has tangential magnitude 6. Multiplying by the circumference 6π gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 + 6(6\pi) = \boxed{36\pi}.$$

Step 2: Direct verification Set $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$, $0 \leq t \leq 2\pi$. Then

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 18,$$

because the radial terms cancel and the tangential terms sum to $18(\sin^2 t + \cos^2 t)$. Therefore

$$\int_0^{2\pi} 18 \, dt = 36\pi,$$

confirming both the magnitude and the positive orientation.