

Question 1

Let C be the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$. Evaluate

$$\int_C y \, dx.$$

Tasks

1. Parametrize the curve using x as the parameter.
2. Replace dx correctly and evaluate the integral.
3. Explain why no arc-length factor appears.

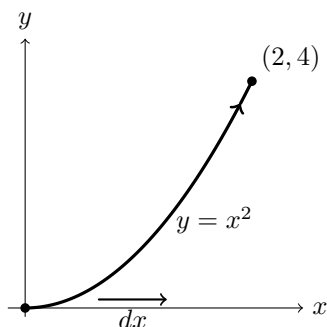
Question 1 – Solution

Strategy. For an integral with respect to x , use $dx = x'(t) dt$ rather than the arc-length element.

Step 1: Parametrize

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad 0 \leq t \leq 2.$$

Thus $x = t$, $y = t^2$, and $dx = dt$.



Step 2: Evaluate

$$\int_C y \, dx = \int_0^2 t^2 \, dt = \left[\frac{t^3}{3} \right]_0^2 = \boxed{\frac{8}{3}}.$$

Step 3: Distinguish from ds The differential is $dx = x'(t) dt = dt$. The speed $\sqrt{1 + 4t^2}$ would be used only for an integral with respect to ds ; inserting it here would change the problem.

Verification Since x increases from 0 to 2 and $0 \leq y \leq 4$, the integral must lie between 0 and $4(2) = 8$. The value $8/3$ satisfies this bound.

Question 2

On the same parabola $C : y = x^2$ from $(0, 0)$ to $(2, 4)$, evaluate

$$\int_C x \, dy.$$

Tasks

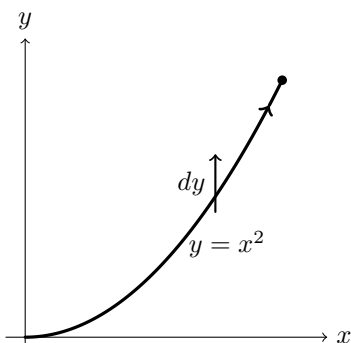
1. Express dy in terms of dx .
2. Evaluate the integral.
3. Compare the answer with $\int_C y \, dx = 8/3$ and explain why they differ.

Question 2 – Solution

Strategy. Use the graph relation to write $dy = y'(x) dx$.

Step 1: Convert the differential Along $y = x^2$,

$$dy = 2x dx, \quad 0 \leq x \leq 2.$$



Step 2: Evaluate

$$\int_C x dy = \int_0^2 x(2x) dx = 2 \left[\frac{x^3}{3} \right]_0^2 = \boxed{\frac{16}{3}}.$$

Step 3: Compare Although both integrals use the same curve, they measure change in different coordinates:

$$y dx = x^2 dx, \quad x dy = x(2x dx) = 2x^2 dx.$$

The second integrand is exactly twice the first, so its value is twice $8/3$.

Verification Here y increases from 0 to 4 and $0 \leq x \leq 2$, so the value must lie between 0 and $2(4) = 8$. The result $16/3$ does.

Question 3

Let C be the straight segment from $A = (0, 0)$ to $B = (2, 1)$. Evaluate

$$\int_C (x + y) dx + (x - y) dy.$$

Tasks

1. Parametrize the segment with the stated orientation.
2. Convert both differentials to the parameter.
3. Evaluate and check the endpoint orientation.

Question 3 – Solution

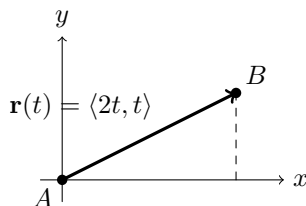
Strategy. A combined differential-form integral becomes one ordinary integral after substituting $dx = x'(t)dt$ and $dy = y'(t)dt$.

Step 1: Parameter data

$$\mathbf{r}(t) = \langle 2t, t \rangle, \quad 0 \leq t \leq 1,$$

so

$$x + y = 3t, \quad x - y = t, \quad dx = 2 dt, \quad dy = dt.$$



Step 2: Evaluate

$$\begin{aligned} \int_C (x + y) dx + (x - y) dy &= \int_0^1 [(3t)(2) + (t)(1)] dt \\ &= \int_0^1 7t dt = \boxed{\frac{7}{2}}. \end{aligned}$$

Verification At $t = 0$ the parametrization gives A , and at $t = 1$ it gives B , so the orientation is correct. Directly grouping the transformed integrand gives $[2(x + y) + (x - y)]dt = (3x + y)dt = 7t dt$, confirming the algebra.

Question 4

Let C_1 be the upper unit semicircle oriented from $(1, 0)$ to $(-1, 0)$, and let C_2 be the same arc with the reverse orientation. Compute

$$\int_C y \, dx$$

for both orientations.

Tasks

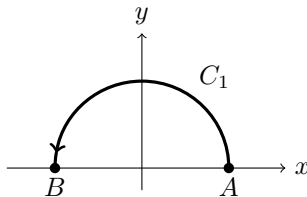
1. Parametrize C_1 and evaluate the integral.
2. Obtain the value on C_2 .
3. Explain why orientation matters for dx but not for ds .

Question 4 – Solution

Strategy. Compute once in the counterclockwise direction, then use the sign change caused by reversing dx .

Step 1: Forward orientation For C_1 , use

$$x = \cos \theta, \quad y = \sin \theta, \quad 0 \leq \theta \leq \pi, \quad dx = -\sin \theta \, d\theta.$$



reversal changes the sign

Therefore

$$\int_{C_1} y \, dx = - \int_0^\pi \sin^2 \theta \, d\theta = \boxed{-\frac{\pi}{2}}.$$

Step 2: Reverse orientation Reversing the same curve negates the coordinate differential, so

$$\boxed{\int_{C_2} y \, dx = \frac{\pi}{2}}.$$

Step 3: Why the rules differ Under reversal, $dx = x'(t)dt$ changes sign because the derivative reverses. By contrast, $ds = |\mathbf{r}'(t)|dt$ uses a magnitude and remains nonnegative.

Verification On C_1 , $y \geq 0$ while x decreases, so $dx \leq 0$ and the integral must be nonpositive. Its reverse must be nonnegative, matching the two answers.

Question 5

Let C be the boundary of the rectangle

$$0 \leq x \leq 3, \quad 0 \leq y \leq 2,$$

traversed counterclockwise. Evaluate

$$\oint_C y \, dx$$

by splitting the boundary into four segments.

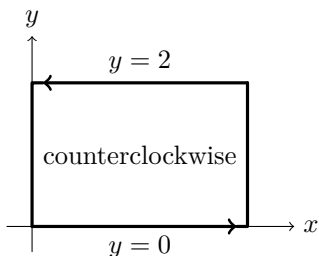
Tasks

1. Determine which edges contribute zero.
2. Evaluate the contribution from each horizontal edge.
3. Add the pieces and check the sign from the orientation.

Question 5 – Solution

Strategy. On vertical edges $dx = 0$, while y is constant on each horizontal edge.

Step 1: Traverse the edges



Along the bottom edge, $y = 0$, so its contribution is 0. On the right and left edges, x is constant, so $dx = 0$ and both contributions are also 0.

Step 2: Top edge Counterclockwise travel moves along the top from $(3, 2)$ to $(0, 2)$. Thus

$$\int_{\text{top}} y \, dx = \int_3^0 2 \, dx = 2[x]_3^0 = -6.$$

Step 3: Sum

$$\oint_C y \, dx = 0 + 0 - 6 + 0 = -6.$$

Verification The only nonzero edge has $y > 0$ and moves left, so $dx < 0$; a negative answer is necessary. Reversing the rectangle would change the answer to $+6$.

Question 6

Let C be the graph $y = x^3$ from $(0, 0)$ to $(1, 1)$. Evaluate

$$\int_C x^2 dy.$$

Tasks

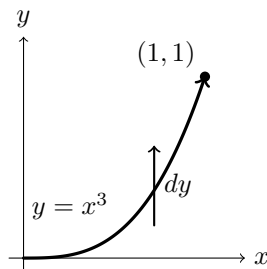
1. Convert dy using the graph equation.
2. Evaluate the resulting single-variable integral.
3. Verify the result by using y as the parameter instead.

Question 6 – Solution

Strategy. First calculate with x as parameter, then reparametrize by y to confirm invariance under a valid change of parameter.

Step 1: Use x

$$y = x^3, \quad dy = 3x^2 dx, \quad 0 \leq x \leq 1.$$



Therefore

$$\int_C x^2 dy = 3 \int_0^1 x^4 dx = 3 \left[\frac{x^5}{5} \right]_0^1 = \boxed{\frac{3}{5}}.$$

Step 2: Use y instead On this interval, $x = y^{1/3}$, so $x^2 = y^{2/3}$. Since the differential is already dy ,

$$\int_C x^2 dy = \int_0^1 y^{2/3} dy = \left[\frac{3}{5} y^{5/3} \right]_0^1 = \frac{3}{5}.$$

Verification Both parametrizations preserve the motion from $(0,0)$ to $(1,1)$ and give the same value, confirming the differential conversion.

Question 7

Let C be the vertical segment $x = 2$ from $(2, 0)$ to $(2, 3)$. Evaluate both

$$\int_C (x + y) \, dx \quad \text{and} \quad \int_C (x + y) \, dy.$$

Tasks

1. Parametrize the vertical segment.
2. Evaluate both integrals.
3. Explain why treating y as a function of x is inappropriate here.

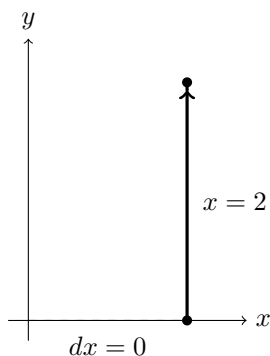
Question 7 – Solution

Strategy. Parametrize by the changing coordinate y ; the constant coordinate has zero differential.

Step 1: Parametrize

$$\mathbf{r}(t) = \langle 2, t \rangle, \quad 0 \leq t \leq 3.$$

Thus $dx = 0$, $dy = dt$, and $x + y = 2 + t$.

**Step 2: Evaluate**

$$\int_C (x + y) dx = 0$$

because $dx = 0$ everywhere. Meanwhile,

$$\int_C (x + y) dy = \int_0^3 (2 + t) dt = \left[2t + \frac{t^2}{2} \right]_0^3 = \boxed{\frac{21}{2}}.$$

Step 3: Representation issue A vertical segment assigns many y -values to the single value $x = 2$, so it cannot be written as a single-valued graph $y = g(x)$. Parametrization avoids this limitation.

Verification The second integrand rises linearly from 2 to 5; its average $7/2$ times the y -interval length 3 gives $21/2$.

Question 8

Let

$$C : \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Evaluate

$$\int_C z \, dx + x \, dy + y \, dz.$$

Tasks

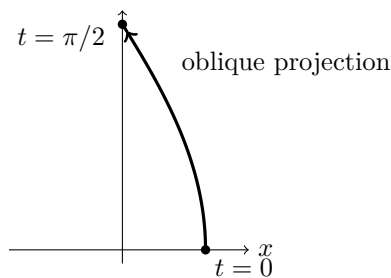
1. Compute dx , dy , and dz .
2. Convert the line integral to one parameter and evaluate it.
3. Verify the cancellation among the three contributions.

Question 8 – Solution

Strategy. Treat each coordinate differential separately, then combine all three ordinary integrals.

Step 1: Differential data

$$\begin{aligned}x &= \cos t, & y &= \sin t, & z &= t, \\dx &= -\sin t \, dt, & dy &= \cos t \, dt, & dz &= dt.\end{aligned}$$



Step 2: Evaluate

$$\int_C z \, dx + x \, dy + y \, dz = \int_0^{\pi/2} (-t \sin t + \cos^2 t + \sin t) \, dt.$$

The three contributions are

$$\int_0^{\pi/2} -t \sin t \, dt = -1, \quad \int_0^{\pi/2} \cos^2 t \, dt = \frac{\pi}{4}, \quad \int_0^{\pi/2} \sin t \, dt = 1.$$

Hence

$$\boxed{\int_C z \, dx + x \, dy + y \, dz = \frac{\pi}{4}}.$$

Verification The first and third contributions cancel exactly, leaving only $\int_0^{\pi/2} \cos^2 t \, dt = \pi/4$.

Question 9

On the parabola $C : y = x^2$ from $(0, 0)$ to $(1, 1)$, a constant a satisfies

$$\int_C ax \, dx + y \, dy = 3.$$

Determine a .

Tasks

1. Convert the entire integral to the parameter x .
2. Solve the resulting equation for a .
3. Substitute the result to verify the specified value.

Question 9 – Solution

Strategy. Use $dy = 2x \, dx$ to make the unknown coefficient appear in a simple algebraic equation.

Step 1: Convert Since $y = x^2$ and $dy = 2x \, dx$,

$$\int_C ax \, dx + y \, dy = \int_0^1 (ax + x^2(2x)) \, dx.$$

Therefore

$$\int_0^1 (ax + 2x^3) \, dx = \frac{a}{2} + \frac{1}{2}.$$

Step 2: Solve

$$\frac{a}{2} + \frac{1}{2} = 3, \quad \frac{a}{2} = \frac{5}{2}, \quad \boxed{a = 5}.$$

Step 3: Verify With $a = 5$,

$$\int_0^1 (5x + 2x^3) \, dx = \left[\frac{5x^2}{2} + \frac{x^4}{2} \right]_0^1 = \frac{5}{2} + \frac{1}{2} = 3.$$

Verification The computed value satisfies the original integral exactly. Because the coefficient of a is $\int_0^1 x \, dx = 1/2 \neq 0$, the solution is unique.

Question 10

Two curves connect $A = (0, 0)$ to $B = (1, 1)$:

$$C_1 : y = x, \quad C_2 : y = x^2, \quad 0 \leq x \leq 1.$$

Compare

$$\int_C y \, dx$$

along the two paths.

Tasks

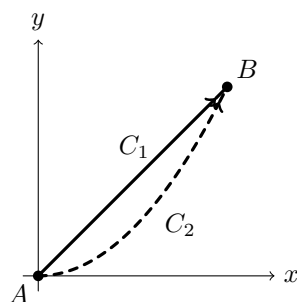
1. Evaluate the integral on C_1 .
2. Evaluate it on C_2 .
3. Explain what the unequal values demonstrate about path dependence.

Question 10 – Solution

Strategy. Use the shared coordinate parameter x on both paths, then compare the scalar functions traced between the same endpoints.

Step 1: Straight path On C_1 , $y = x$, so

$$\int_{C_1} y \, dx = \int_0^1 x \, dx = \boxed{\frac{1}{2}}.$$



Step 2: Parabolic path On C_2 , $y = x^2$, so

$$\int_{C_2} y \, dx = \int_0^1 x^2 \, dx = \boxed{\frac{1}{3}}.$$

Step 3: Interpretation Both curves have the same initial point, terminal point, and orientation, yet

$$\frac{1}{2} \neq \frac{1}{3}.$$

Therefore the value of $\int_C y \, dx$ is not determined by endpoints alone; it depends on the path between them.

Verification For $0 < x < 1$, the line $y = x$ lies above the parabola $y = x^2$. With $dx > 0$, its integral must be larger, consistent with $1/2 > 1/3$.