

Question 1

Evaluate the scalar line integral

$$\int_C (x + y) \, ds,$$

where C is the line segment from $(0, 0)$ to $(3, 4)$.

Tasks

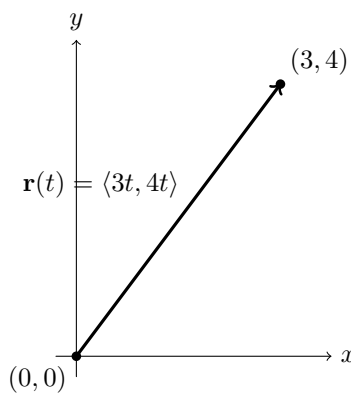
1. Parametrize C on $0 \leq t \leq 1$.
2. Compute ds and evaluate the integral.
3. Verify the constant speed and total length of the segment.

Question 1 – Solution

Strategy. Use linear interpolation between the endpoints, then replace ds by $|\mathbf{r}'(t)| dt$.

Step 1: Parametrize the segment

$$\mathbf{r}(t) = \langle 3t, 4t \rangle, \quad 0 \leq t \leq 1.$$



Along the curve,

$$x + y = 3t + 4t = 7t.$$

Step 2: Arc-length element

$$\mathbf{r}'(t) = \langle 3, 4 \rangle, \quad |\mathbf{r}'(t)| = 5, \quad ds = 5 dt.$$

Therefore

$$\int_C (x + y) ds = \int_0^1 (7t)(5) dt = 35 \left[\frac{t^2}{2} \right]_0^1 = \boxed{\frac{35}{2}}.$$

Verification The speed is constant and the parameter interval has length 1, so the curve length is 5, agreeing with the distance formula $\sqrt{3^2 + 4^2} = 5$. The integrand rises linearly from 0 to 7, whose average is 7/2; multiplying by length 5 again gives 35/2.

Question 2

Evaluate

$$\int_C x \, ds,$$

where C is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Tasks

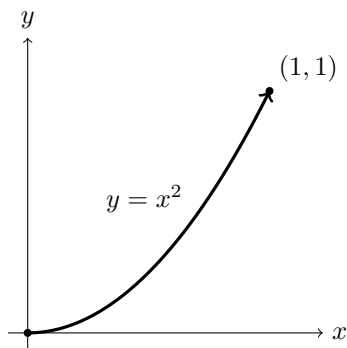
1. Use x itself as a parameter.
2. Derive the arc-length element and evaluate the integral.
3. Check the sign and size of the result.

Question 2 – Solution

Strategy. Parametrize the graph by $\mathbf{r}(x) = \langle x, x^2 \rangle$ so the given scalar function is already the parameter.

Step 1: Parameter and speed

$$\begin{aligned}\mathbf{r}(x) &= \langle x, x^2 \rangle, & 0 \leq x \leq 1, \\ \mathbf{r}'(x) &= \langle 1, 2x \rangle, & ds = \sqrt{1 + 4x^2} \, dx.\end{aligned}$$



Step 2: Evaluate Let $u = 1 + 4x^2$, so $du = 8x \, dx$. Then

$$\begin{aligned}\int_C x \, ds &= \int_0^1 x \sqrt{1 + 4x^2} \, dx \\ &= \frac{1}{8} \int_1^5 u^{1/2} \, du = \frac{1}{12} \left[u^{3/2} \right]_1^5 = \boxed{\frac{5\sqrt{5} - 1}{12}}.\end{aligned}$$

Verification Both x and ds are nonnegative, so the answer must be positive. Also $0 \leq x \leq 1$ implies the integral is no larger than the curve length; numerically the result is about 0.848, consistent with that bound.

Question 3

Let C be the upper semicircle

$$x^2 + y^2 = 4, \quad y \geq 0,$$

oriented from $(2, 0)$ to $(-2, 0)$. Evaluate

$$\int_C (x^2 + y^2) \, ds.$$

Tasks

1. Parametrize the semicircle with its stated orientation.
2. Evaluate the line integral.
3. Give a geometric verification without integration.

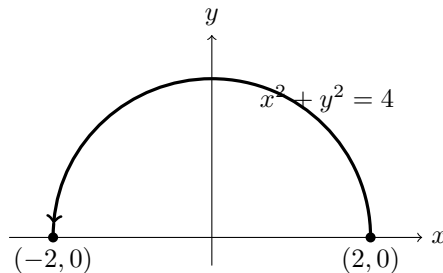
Question 3 – Solution

Strategy. On a circle, the scalar integrand $x^2 + y^2$ is constant, so the integral is that constant times arc length.

Step 1: Parametrization

$$\mathbf{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta \rangle, \quad 0 \leq \theta \leq \pi.$$

This begins at $(2, 0)$ and ends at $(-2, 0)$, as required.

**Step 2: Evaluate** Along C ,

$$x^2 + y^2 = 4, \quad |\mathbf{r}'(\theta)| = | \langle -2 \sin \theta, 2 \cos \theta \rangle | = 2.$$

Thus

$$\int_C (x^2 + y^2) ds = \int_0^\pi (4)(2) d\theta = \boxed{8\pi}.$$

Verification The upper semicircle has length $\frac{1}{2}(2\pi \cdot 2) = 2\pi$. Since the integrand is constantly 4, the integral must be $4(2\pi) = 8\pi$.

Question 4

Let C be the helix

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle, \quad 0 \leq t \leq \pi.$$

Evaluate

$$\int_C (x^2 + y^2 + z) \, ds.$$

Tasks

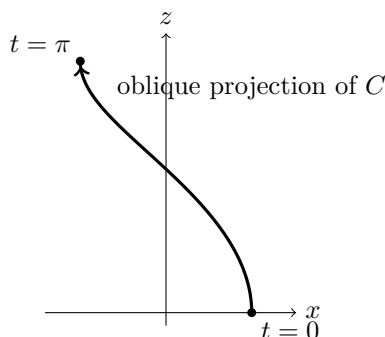
1. Express the scalar function along the helix.
2. Compute the speed and evaluate the integral.
3. Verify the result by separating the constant and variable parts.

Question 4 – Solution

Strategy. Substitute the parametrization into both the scalar function and the arc-length factor.

Step 1: Restrict the integrand Along C ,

$$x^2 + y^2 + z = 4 \cos^2 t + 4 \sin^2 t + t = 4 + t.$$



Step 2: Arc length and integral

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad |\mathbf{r}'(t)| = \sqrt{4 + 1} = \sqrt{5}.$$

Therefore

$$\begin{aligned} \int_C (x^2 + y^2 + z) ds &= \sqrt{5} \int_0^\pi (4 + t) dt \\ &= \sqrt{5} \left[4t + \frac{t^2}{2} \right]_0^\pi = \boxed{\sqrt{5} \left(4\pi + \frac{\pi^2}{2} \right)}. \end{aligned}$$

Verification The constant part 4 contributes $4L = 4\pi\sqrt{5}$ because $L = \pi\sqrt{5}$. The $z = t$ part has average $\pi/2$ and contributes $(\pi/2)L = \pi^2\sqrt{5}/2$, matching the two terms above.

Question 5

The piecewise smooth curve $C = C_1 \cup C_2$ travels from

$$A = (0, 0) \quad \text{to} \quad B = (1, 0) \quad \text{to} \quad D = (1, 2)$$

along straight segments. Evaluate

$$\int_C (xy + 1) \, ds.$$

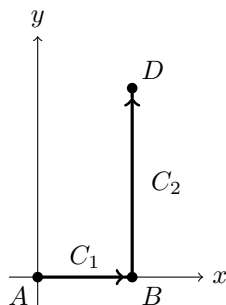
Tasks

1. Parametrize each segment.
2. Evaluate the integral on each piece and add the results.
3. Check the contribution using the integrand along each segment.

Question 5 – Solution

Strategy. A scalar line integral over a piecewise curve is the sum of the integrals over its smooth pieces.

Step 1: First segment



For C_1 , use $\mathbf{r}_1(t) = \langle t, 0 \rangle$, $0 \leq t \leq 1$. Then $ds = dt$ and $xy + 1 = 1$, so

$$\int_{C_1} (xy + 1) ds = \int_0^1 1 dt = 1.$$

Step 2: Second segment For C_2 , use $\mathbf{r}_2(u) = \langle 1, u \rangle$, $0 \leq u \leq 2$. Then $ds = du$ and $xy + 1 = u + 1$, giving

$$\int_{C_2} (xy + 1) ds = \int_0^2 (u + 1) du = \left[\frac{u^2}{2} + u \right]_0^2 = 4.$$

Hence

$$\boxed{\int_C (xy + 1) ds = 1 + 4 = 5}.$$

Verification On C_1 the constant value 1 times length 1 gives 1. On C_2 the integrand rises linearly from 1 to 3, so its average 2 times length 2 gives 4.

Question 6

Let C be the unit circle. Compute

$$\int_C (x + 2) \, ds$$

once with a counterclockwise parametrization and once with a clockwise parametrization.

Tasks

1. Write parametrizations for both orientations.
2. Show directly that both integrals have the same value.
3. Explain why every scalar line integral with respect to ds is orientation-independent.

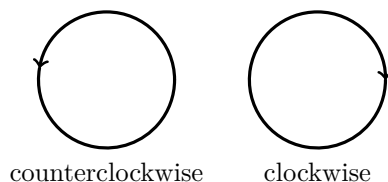
Question 6 – Solution

Strategy. Reversing direction changes the parameter derivative's sign but not its magnitude, so ds is unchanged.

Step 1: Counterclockwise Let $\mathbf{r}_+(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq 2\pi$. Since $|\mathbf{r}'_+(t)| = 1$,

$$I_+ = \int_0^{2\pi} (\cos t + 2) dt = 0 + 4\pi = 4\pi.$$

same unit circle, opposite orientations



Step 2: Clockwise Let $\mathbf{r}_-(t) = \langle \cos t, -\sin t \rangle$, $0 \leq t \leq 2\pi$. Again the speed is 1, and $x = \cos t$, so

$$I_- = \int_0^{2\pi} (\cos t + 2) dt = \boxed{4\pi}.$$

Step 3: General reason Under a reversed parametrization, $\tilde{\mathbf{r}}(t) = \mathbf{r}(a+b-t)$, the derivative is $-\mathbf{r}'(a+b-t)$, but

$$|\tilde{\mathbf{r}}'(t)| = |\mathbf{r}'(a+b-t)|.$$

After substitution, the scalar values and arc-length factors are identical. Therefore $\int_C f ds$ does not depend on orientation.

Verification Here the x -term cancels by symmetry, while the constant 2 contributes 2 times the circumference 2π , confirming 4π .

Question 7

A wire follows the first-quadrant astroid arc

$$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t \rangle, \quad 0 \leq t \leq \frac{\pi}{2},$$

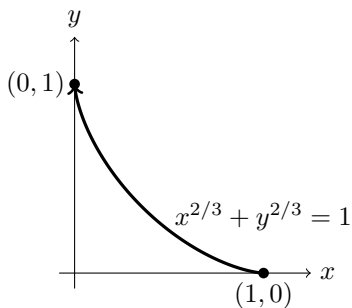
and has linear density $\delta(x, y) = x + y$. Find its mass.

Tasks

1. Derive and simplify the speed on the given interval.
2. Express the density in terms of t and evaluate the mass.
3. Check positivity and endpoint behavior.

Question 7 – Solution

Strategy. Differentiate the astroid parametrization; in the first quadrant the absolute values in the speed simplify.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -3 \cos^2 t \sin t, 3 \sin^2 t \cos t \rangle.$$

Because $\sin t, \cos t \geq 0$,

$$\begin{aligned} |\mathbf{r}'(t)| &= 3 \sin t \cos t \sqrt{\cos^2 t + \sin^2 t} \\ &= 3 \sin t \cos t. \end{aligned}$$

Step 2: Mass Along the wire, $\delta = \cos^3 t + \sin^3 t$. Hence

$$\begin{aligned} M &= \int_0^{\pi/2} 3(\cos^3 t + \sin^3 t) \sin t \cos t \, dt \\ &= 3 \left(\int_0^{\pi/2} \cos^4 t \sin t \, dt + \int_0^{\pi/2} \sin^4 t \cos t \, dt \right) \\ &= 3 \left(\frac{1}{5} + \frac{1}{5} \right) = \boxed{\frac{6}{5}}. \end{aligned}$$

Verification Density and speed are nonnegative. The speed vanishes only at the two cusps, which does not affect the integral; the finite positive mass $6/5$ is therefore consistent.

Question 8

Let C be the circle $x^2 + y^2 = a^2$, where $a > 0$. Find the average value of

$$f(x, y) = x^2$$

along C with respect to arc length.

Tasks

1. Compute $\int_C x^2 ds$ and the length of C .
2. Find the arc-length average value.
3. Identify all points on C where x^2 equals that average.

Question 8 – Solution

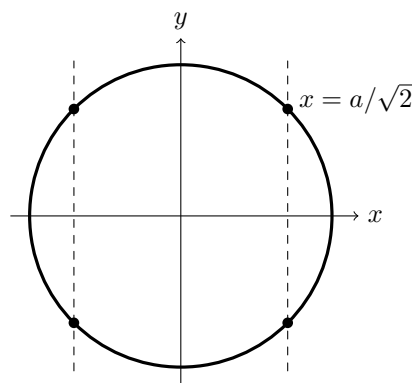
Strategy. Parametrize the circle and divide the scalar line integral by total arc length.

Step 1: Integral and length Use

$$\mathbf{r}(\theta) = \langle a \cos \theta, a \sin \theta \rangle, \quad 0 \leq \theta \leq 2\pi.$$

Then $ds = a d\theta$, so

$$\int_C x^2 ds = a^3 \int_0^{2\pi} \cos^2 \theta d\theta = \pi a^3, \quad L = 2\pi a.$$



$$x^2 + y^2 = a^2 \text{ (scaled sketch)}$$

Step 2: Average value

$$f_{\text{avg}} = \frac{1}{L} \int_C f ds = \frac{\pi a^3}{2\pi a} = \boxed{\frac{a^2}{2}}.$$

Step 3: Points attaining the average We need $x^2 = a^2/2$, so $x = \pm a/\sqrt{2}$. The circle equation then gives $y = \pm a/\sqrt{2}$, independently. Thus the four points are

$$\boxed{\left(\frac{\pm a}{\sqrt{2}}, \frac{\pm a}{\sqrt{2}} \right)},$$

with all four sign combinations.

Verification Symmetry gives equal averages for x^2 and y^2 . Their sum is constantly a^2 on C , so each average must be $a^2/2$.

Question 9

For $a > 0$, a straight wire runs from $(0, 0)$ to (a, a) and has density

$$\delta(x, y) = x + y.$$

Its mass is $8\sqrt{2}$. Determine a and the wire's length.

Tasks

1. Parametrize the wire and derive its mass as a function of a .
2. Solve the inverse condition, respecting $a > 0$.
3. Compute the resulting length and verify the mass.

Question 9 – Solution

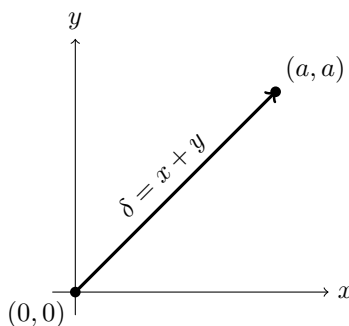
Strategy. Express the line integral entirely in terms of the unknown endpoint coordinate a , then solve the resulting quadratic equation.

Step 1: Mass formula Use

$$\mathbf{r}(t) = \langle at, at \rangle, \quad 0 \leq t \leq 1.$$

Then $\delta = 2at$ and $ds = |\langle a, a \rangle| dt = a\sqrt{2} dt$. Therefore

$$M(a) = \int_0^1 (2at)(a\sqrt{2}) dt = 2a^2\sqrt{2} \left[\frac{t^2}{2} \right]_0^1 = a^2\sqrt{2}.$$



Step 2: Recover a The given mass implies

$$a^2\sqrt{2} = 8\sqrt{2}, \quad a^2 = 8.$$

Because $a > 0$,

$$\boxed{a = 2\sqrt{2}}.$$

Step 3: Length and verification The diagonal length is

$$L = \sqrt{a^2 + a^2} = a\sqrt{2} = (2\sqrt{2})(\sqrt{2}) = \boxed{4}.$$

Substitution into the mass formula gives $M = (2\sqrt{2})^2\sqrt{2} = 8\sqrt{2}$, exactly the specified mass.

Question 10

A wire occupies the upper semicircle

$$x^2 + y^2 = R^2, \quad y \geq 0,$$

and has density $\delta(x, y) = 1 + y/R$, where $R > 0$.

Tasks

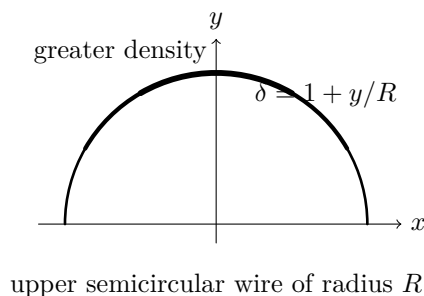
1. Find the wire's mass.
2. Compute its moments about the x - and y -axes.
3. Determine its center of mass and check the result by symmetry and bounds.

Question 10 – Solution

Strategy. Parametrize by polar angle; symmetry immediately predicts that the center of mass lies on the y -axis.

Step 1: Mass Let $x = R \cos \theta$, $y = R \sin \theta$, $0 \leq \theta \leq \pi$, so $ds = R d\theta$ and $\delta = 1 + \sin \theta$. Then

$$M = R \int_0^\pi (1 + \sin \theta) d\theta = \boxed{R(\pi + 2)}.$$



Step 2: Moments The moment about the x -axis is

$$\begin{aligned} M_x &= \int_C y \delta ds = R^2 \int_0^\pi \sin \theta (1 + \sin \theta) d\theta \\ &= R^2 \left(2 + \frac{\pi}{2} \right). \end{aligned}$$

For the y -axis,

$$M_y = R^2 \int_0^\pi \cos \theta (1 + \sin \theta) d\theta = 0.$$

Step 3: Center of mass

$$\bar{x} = \frac{M_y}{M} = 0, \quad \bar{y} = \frac{M_x}{M} = \boxed{\frac{R(\pi + 4)}{2(\pi + 2)}}.$$

Thus

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{R(\pi + 4)}{2(\pi + 2)} \right)}.$$

Verification Left-right symmetry forces $\bar{x} = 0$. Also $0 < (\pi + 4)/[2(\pi + 2)] < 1$, so $0 < \bar{y} < R$, placing the center of mass inside the semicircle's vertical range.