

Question 1

Consider the planar vector field

$$\mathbf{F}(x, y) = \langle x, -y \rangle.$$

Tasks

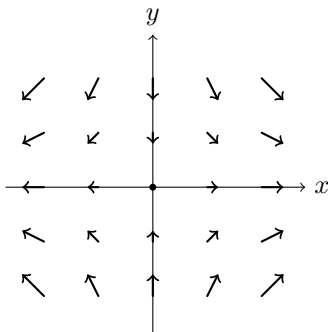
1. Evaluate $\mathbf{F}$ at $(1, 2)$, $(-2, 1)$, and $(0, 0)$.
2. Find the magnitude and direction of $\mathbf{F}(1, 2)$.
3. Describe the field on the coordinate axes and identify every zero vector.

Question 1 – Solution

Strategy. Evaluate the two components pointwise, then use the component signs to understand the field globally.

Step 1: Sample vectors

$$\mathbf{F}(1, 2) = \langle 1, -2 \rangle, \quad \mathbf{F}(-2, 1) = \langle -2, -1 \rangle, \quad \mathbf{F}(0, 0) = \langle 0, 0 \rangle.$$



$$\mathbf{F}(x, y) = \langle x, -y \rangle$$

Step 2: Magnitude and direction At $(1, 2)$,

$$|\mathbf{F}| = \sqrt{1^2 + (-2)^2} = \boxed{\sqrt{5}}, \quad \hat{\mathbf{F}} = \boxed{\left\langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle}.$$

Step 3: Axis behavior and zeros On the x -axis, $\mathbf{F}(x, 0) = \langle x, 0 \rangle$, so arrows point away from the origin. On the y -axis, $\mathbf{F}(0, y) = \langle 0, -y \rangle$, so arrows point toward the origin. Finally,

$$\mathbf{F}(x, y) = \mathbf{0} \iff x = 0 \text{ and } y = 0.$$

Thus the origin is the field's only zero.

Verification The plotted arrows reverse horizontally when x changes sign and vertically when y changes sign, exactly matching the formula.

Question 2

Determine the domain of the vector field

$$\mathbf{F}(x, y) = \left\langle \ln(x - y), \sqrt{4 - x^2 - y^2} \right\rangle.$$

Tasks

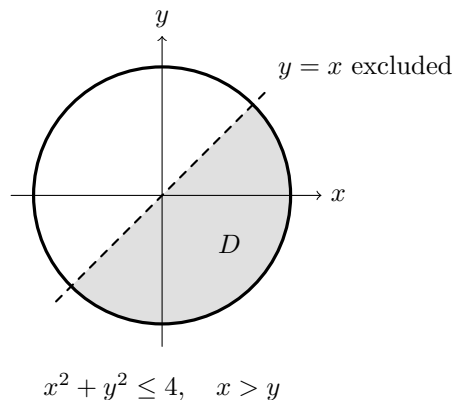
1. Express the domain as simultaneous inequalities.
2. Decide which parts of the line $y = x$ and circle $x^2 + y^2 = 4$ are included.
3. State where the field is continuous and where both components are differentiable.

Question 2 – Solution

Strategy. Intersect the natural domains of the logarithm and square root, keeping track of strict versus non-strict inequalities.

Step 1: Component restrictions The logarithm requires $x - y > 0$, while the square root requires $4 - x^2 - y^2 \geq 0$. Therefore

$$D = \{(x, y) : x > y, x^2 + y^2 \leq 4\}.$$



Step 2: Boundary decisions No point of $y = x$ is included because $\ln 0$ is undefined. The circular boundary is included only where $x > y$; there the second component equals 0 and the first remains defined.

Step 3: Regularity Both component functions are continuous wherever they are defined, so $\mathbf{F}$ is continuous on all of D . Both are differentiable where their defining inequalities are strict:

$$x > y, \quad x^2 + y^2 < 4.$$

At the included circular arc, the square-root component is continuous but its derivatives become singular.

Verification The shaded half-disk excludes its straight diagonal edge but includes the appropriate circular arc, reflecting $>$ versus $\leq$.

Question 3

For

$$\mathbf{F}(x, y) = \frac{\langle x, y \rangle}{x^2 + y^2},$$

analyze the field geometrically.

Tasks

1. State its domain and find its magnitude at a point of radius r .
2. Determine its direction and compare vectors at $(1, 0)$, $(2, 0)$, and $(0, -2)$.
3. Describe what happens as r approaches 0 and as r grows.

Question 3 – Solution

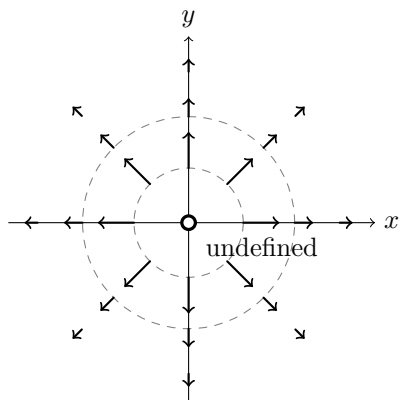
Strategy. Factor the position vector into its length and radial unit vector.

Step 1: Polar form With $r = \sqrt{x^2 + y^2}$ and $\mathbf{e}_r = \langle x/r, y/r \rangle$,

$$\mathbf{F} = \frac{r\mathbf{e}_r}{r^2} = \frac{1}{r}\mathbf{e}_r.$$

Hence the domain is $\mathbb{R}^2 \setminus \{(0, 0)\}$, the direction is radially outward, and

$$|\mathbf{F}| = \frac{1}{r}.$$



outward direction; magnitude $1/r$

Step 2: Samples

$$\mathbf{F}(1, 0) = \langle 1, 0 \rangle, \quad \mathbf{F}(2, 0) = \left\langle \frac{1}{2}, 0 \right\rangle, \quad \mathbf{F}(0, -2) = \left\langle 0, -\frac{1}{2} \right\rangle.$$

Doubling the radius halves the arrow length while preserving radial direction.

Step 3: Limiting behavior As $r \rightarrow 0^+$, $|\mathbf{F}| = 1/r \rightarrow \infty$, explaining the missing origin. As $r \rightarrow \infty$, the magnitude tends to 0.

Verification Multiplying any vector by its radius gives $\mathbf{e}_r$, a unit vector, so both the direction and inverse-radius magnitude are confirmed.

Question 4

A three-dimensional gravitational field is modeled by

$$\mathbf{G}(x, y, z) = -12 \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{3/2}}.$$

Tasks

1. Evaluate $\mathbf{G}$ at $P = (2, -1, 2)$.
2. Find its magnitude and a unit vector in its direction.
3. Explain the minus sign and state the field's domain.

Question 4 – Solution

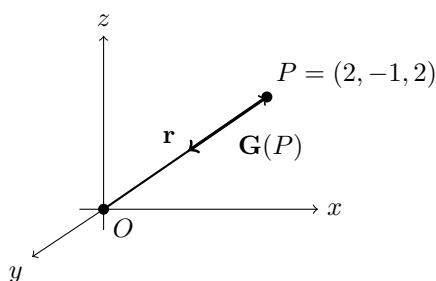
Strategy. Compute the distance from the origin first; it controls both the denominator and the inverse-square magnitude.

Step 1: Evaluate at the point At $P = (2, -1, 2)$,

$$r = \sqrt{2^2 + (-1)^2 + 2^2} = 3, \quad r^3 = 27.$$

Therefore

$$\mathbf{G}(P) = -\frac{12}{27}\langle 2, -1, 2 \rangle = \boxed{\left\langle -\frac{8}{9}, \frac{4}{9}, -\frac{8}{9} \right\rangle}.$$



Step 2: Magnitude and unit direction In general, $|\mathbf{G}| = 12r/r^3 = 12/r^2$. Thus

$$|\mathbf{G}(P)| = \boxed{\frac{4}{3}}, \quad \widehat{\mathbf{G}(P)} = \boxed{\left\langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle}.$$

Step 3: Interpretation The negative sign reverses the outward position direction, so gravity points toward the origin. The denominator vanishes only at the origin; hence the domain is

$$\boxed{\mathbb{R}^3 \setminus \{(0, 0, 0)\}}.$$

Verification The displayed unit vector has length 1, and multiplying it by $4/3$ reproduces $\mathbf{G}(P)$.

Question 5

Let

$$\mathbf{F}(x, y) = \langle -y, x \rangle.$$

Tasks

1. Prove that $\mathbf{F}$ is tangent to every circle $x^2 + y^2 = c^2$ centered at the origin.
2. Find its magnitude and orientation.
3. At $P = (3, 4)$, find the vector and the corresponding unit tangent direction.

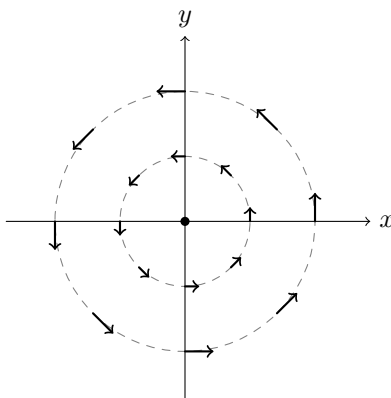
Question 5 – Solution

Strategy. A circle's radial vector is normal to the circle, so orthogonality to that vector proves tangency.

Step 1: Tangency At (x, y) the radial normal is $\mathbf{r} = \langle x, y \rangle$. Since

$$\mathbf{F} \cdot \mathbf{r} = \langle -y, x \rangle \cdot \langle x, y \rangle = -xy + xy = 0,$$

$\mathbf{F}$ is tangent to every centered circle.



vectors are tangent to centered circles

Step 2: Magnitude and orientation

$$|\mathbf{F}| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2 + y^2} = r.$$

At $(r, 0)$ the vector is $\langle 0, r \rangle$, which points upward; therefore the orientation is counterclockwise.

Step 3: The point $(3, 4)$

$$\mathbf{F}(3, 4) = \langle -4, 3 \rangle, \quad |\mathbf{F}(3, 4)| = 5,$$

so the counterclockwise unit tangent is

$$\left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle.$$

Verification The dot product $\langle -4, 3 \rangle \cdot \langle 3, 4 \rangle = -12 + 12 = 0$, confirming tangency at P .

Question 6

A temperature distribution is

$$T(x, y) = 100 - x^2 - 4y^2.$$

The heat-flux vector field is $\mathbf{H} = -\frac{1}{2}\nabla T$.

Tasks

1. Find a formula for $\mathbf{H}(x, y)$.
2. Evaluate its magnitude and unit direction at $P = (2, 1)$.
3. Show that the heat flux is normal to the isotherm through P and identify that isotherm.

Question 6 – Solution

Strategy. The gradient is normal to level curves; multiplying it by a negative constant reverses direction but preserves normality.

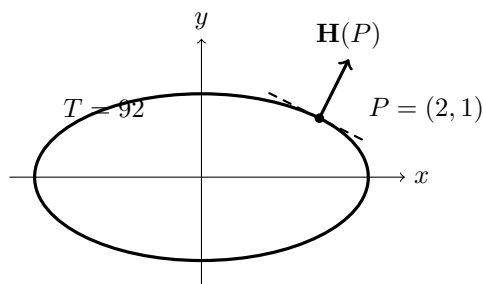
Step 1: Construct the field

$$\nabla T = \langle -2x, -8y \rangle, \quad \mathbf{H} = -\frac{1}{2}\nabla T = \boxed{\langle x, 4y \rangle}.$$

Step 2: Evaluate at P

$$\mathbf{H}(2, 1) = \langle 2, 4 \rangle, \quad |\mathbf{H}(2, 1)| = \boxed{2\sqrt{5}},$$

$$\widehat{\mathbf{H}}(2, 1) = \boxed{\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle}.$$



heat flux is normal to the isotherm

Step 3: Isotherm and normality Since $T(2, 1) = 100 - 4 - 4 = 92$, the isotherm is

$$x^2 + 4y^2 = 8.$$

Its normal direction is $\nabla(x^2 + 4y^2) = \langle 2x, 8y \rangle$. At P this is $\langle 4, 8 \rangle = 2\mathbf{H}(P)$, so the flux is normal to the ellipse and points toward lower temperature.

Verification Moving a short distance along $\mathbf{H}(P)$ increases $x^2 + 4y^2$, so it decreases $T = 100 - (x^2 + 4y^2)$, as heat flow should.

Question 7

A linear planar vector field has the form

$$\mathbf{F}(x, y) = \langle ax + by, cx + dy \rangle.$$

Measurements give

$$\mathbf{F}(1, 0) = \langle 2, -1 \rangle, \quad \mathbf{F}(0, 1) = \langle 3, 4 \rangle.$$

Tasks

1. Determine a, b, c, d and write the field explicitly.
2. Evaluate $\mathbf{F}(-1, 2)$.
3. Find every point where $\mathbf{F} = \mathbf{0}$ and explain uniqueness.

Question 7 – Solution

Strategy. The values on the coordinate basis vectors are exactly the columns of the matrix representing a linear field.

Step 1: Recover the coefficients Substitution at $(1, 0)$ gives $\langle a, c \rangle = \langle 2, -1 \rangle$, while substitution at $(0, 1)$ gives $\langle b, d \rangle = \langle 3, 4 \rangle$. Hence

$$\mathbf{F}(x, y) = \langle 2x + 3y, -x + 4y \rangle.$$

Equivalently,

$$\mathbf{F}(x, y) = \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Step 2: Evaluate

$$\mathbf{F}(-1, 2) = \langle -2 + 6, 1 + 8 \rangle = \langle 4, 9 \rangle.$$

Step 3: Find zero vectors Solve

$$2x + 3y = 0, \quad -x + 4y = 0.$$

The second equation gives $x = 4y$; the first then gives $11y = 0$. Thus

$$(x, y) = (0, 0)$$

is the only zero. Equivalently, the coefficient determinant is $2(4) - 3(-1) = 11 \neq 0$, so the linear map has a trivial nullspace.

Verification The recovered formula returns both measured vectors when the two basis points are substituted, so all four coefficients satisfy the data.

Question 8

For a real parameter λ , consider

$$\mathbf{F}_\lambda(x, y) = \langle x - y, \lambda x + y \rangle.$$

Tasks

1. Find all equilibrium points satisfying $\mathbf{F}_\lambda = \mathbf{0}$.
2. Identify the exceptional value of λ and describe its equilibrium set.
3. For that exceptional value, describe the field on either side of the equilibrium set.

Question 8 – Solution

Strategy. Solve the two component equations and separate the parameter value that makes them dependent.

Step 1: Solve generally The first component gives $x = y$. Substituting into the second gives

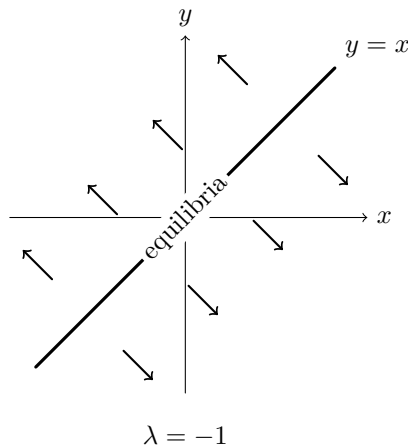
$$\lambda x + y = (\lambda + 1)x = 0.$$

If $\lambda \neq -1$, then $x = 0$ and $y = 0$, so

$$\lambda \neq -1 : \text{the origin is the only equilibrium}.$$

Step 2: Exceptional value If $\lambda = -1$, the second component is $-x + y$, the negative of the first. Every point with $x = y$ is then a zero:

$$\lambda = -1 : \{(x, y) : y = x\} \text{ is the equilibrium set}.$$



Step 3: Behavior off the line For $\lambda = -1$, set $d = x - y$. Then

$$\mathbf{F}_{-1} = \langle d, -d \rangle = d\langle 1, -1 \rangle.$$

This vector is normal to $y = x$. When $x > y$, it points southeast; when $x < y$, it points northwest. Both directions lead away from the equilibrium line.

Verification The coefficient determinant is $1(1) - (-1)\lambda = 1 + \lambda$, which vanishes exactly at $\lambda = -1$, confirming the case split.

Question 9

Let

$$\mathbf{F}(x, y, z) = \langle y, z, x \rangle$$

and let the helix be

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$

Tasks

1. Express the field along the helix as $\mathbf{F}(\mathbf{r}(t))$.
2. Find the unit tangent $\mathbf{T}(t)$.
3. At $t = 0$, decompose $\mathbf{F}(\mathbf{r}(0))$ into components parallel and perpendicular to the helix.

Question 9 – Solution

Strategy. Substitute the curve into the field, then use vector projection onto the unit tangent.

Step 1: Restrict the field to the curve

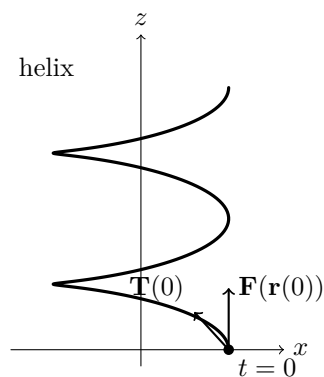
$$\mathbf{F}(\mathbf{r}(t)) = \langle \sin t, t, \cos t \rangle.$$

Also,

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle, \quad |\mathbf{r}'(t)| = \sqrt{2},$$

so

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.$$



Step 2: Data at $t = 0$

$$\mathbf{F}(\mathbf{r}(0)) = \langle 0, 0, 1 \rangle, \quad \mathbf{T}(0) = \frac{1}{\sqrt{2}} \langle 0, 1, 1 \rangle.$$

The scalar tangential component is

$$\mathbf{F} \cdot \mathbf{T} = \frac{1}{\sqrt{2}}.$$

Step 3: Vector decomposition

$$\mathbf{F}_{\parallel} = (\mathbf{F} \cdot \mathbf{T})\mathbf{T} = \left\langle 0, \frac{1}{2}, \frac{1}{2} \right\rangle,$$

$$\mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{\parallel} = \left\langle 0, -\frac{1}{2}, \frac{1}{2} \right\rangle.$$

Verification The two boxed vectors sum to $\langle 0, 0, 1 \rangle$, and $\mathbf{F}_{\perp} \cdot \mathbf{T}(0) = 0$.

Question 10

On the unit circle $x^2 + y^2 = 1$, consider

$$\mathbf{F}(x, y) = \langle 2x - y, x + 2y \rangle.$$

Tasks

1. Decompose $\mathbf{F}$ into outward radial and counterclockwise tangential components.
2. Find its magnitude and its angle from the outward radial direction.
3. Evaluate the decomposition at $(0, 1)$ and determine whether the field has any zero away from the origin.

Question 10 – Solution

Strategy. Recognize the field as a fixed combination of the radial vector and its 90° counterclockwise rotation.

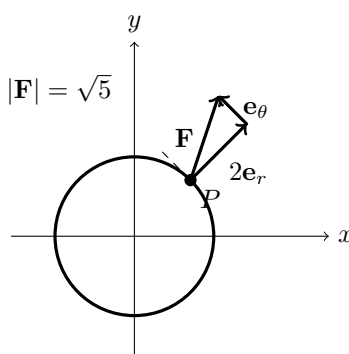
Step 1: Decompose On the unit circle,

$$\mathbf{e}_r = \langle x, y \rangle, \quad \mathbf{e}_\theta = \langle -y, x \rangle.$$

Therefore

$$\mathbf{F} = 2\langle x, y \rangle + \langle -y, x \rangle = \boxed{2\mathbf{e}_r + \mathbf{e}_\theta}.$$

The radial scalar component is 2 and the counterclockwise tangential scalar component is 1.



Step 2: Magnitude and angle Because the two unit vectors are perpendicular,

$$\boxed{|\mathbf{F}| = \sqrt{2^2 + 1^2} = \sqrt{5}}.$$

If α is measured counterclockwise from $\mathbf{e}_r$, then

$$\boxed{\alpha = \tan^{-1}\left(\frac{1}{2}\right)}.$$

Step 3: Point check and zeros At $(0, 1)$,

$$\mathbf{e}_r = \langle 0, 1 \rangle, \quad \mathbf{e}_\theta = \langle -1, 0 \rangle, \quad \mathbf{F} = \boxed{\langle -1, 2 \rangle}.$$

For the full plane, $\mathbf{F} = \mathbf{0}$ gives $2x - y = 0$ and $x + 2y = 0$. Their coefficient determinant is $5 \neq 0$, so only the origin is a zero; in particular, there are none on the unit circle.

Verification The sample vector $\langle -1, 2 \rangle$ has length $\sqrt{5}$ and equals $2\mathbf{e}_r + \mathbf{e}_\theta$, confirming all parts.