

Question 1

Find the area of the part of the plane

$$z = 2x - y$$

lying above the rectangle $0 \leq x \leq 3$, $0 \leq y \leq 2$.

Tasks

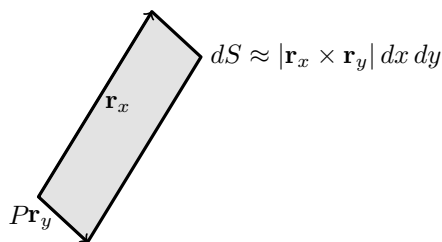
1. Derive the graph surface-area factor from tangent vectors.
2. Evaluate the surface area.
3. Verify by interpreting the constant area scale.

Question 1 – Solution

Strategy. Parametrize the graph by its projection and use the tangent-parallelogram area scale.

Step 1: Derive the element For a graph $z = f(x, y)$,

$$\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle, \quad \mathbf{r}_x = \langle 1, 0, f_x \rangle, \quad \mathbf{r}_y = \langle 0, 1, f_y \rangle.$$



Thus

$$|\mathbf{r}_x \times \mathbf{r}_y| = |\langle -f_x, -f_y, 1 \rangle| = \sqrt{1 + f_x^2 + f_y^2}.$$

Step 2: Apply to the plane Here $f_x = 2$ and $f_y = -1$, so

$$S = \int_0^3 \int_0^2 \sqrt{1 + 2^2 + (-1)^2} dy dx = \sqrt{6}(3)(2) = \boxed{6\sqrt{6}}.$$

Verification Every small projected area is enlarged by the same factor $\sqrt{6}$. Since the rectangle has area 6, the surface area must be $6\sqrt{6}$.

Question 2

Find the area of the paraboloid

$$z = x^2 + y^2$$

above the disk $x^2 + y^2 \leq 4$.

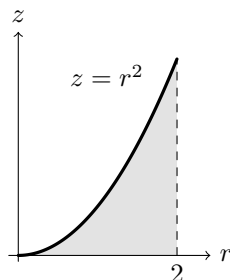
Tasks

1. Form the graph surface-area integrand.
2. Use polar coordinates to evaluate the integral.
3. Check the answer against the projected area.

Question 2 – Solution

Strategy. The squared gradient depends only on r , so polar coordinates reduce the area integral to one variable.

Step 1: Geometry and gradient



For $f = x^2 + y^2$,

$$f_x = 2x, \quad f_y = 2y, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4r^2}.$$

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} \, r \, dr \, d\theta \\ &= 2\pi \left[\frac{(1 + 4r^2)^{3/2}}{12} \right]_0^2 = \boxed{\frac{\pi}{6} (17\sqrt{17} - 1)}. \end{aligned}$$

Verification The area factor is at least 1, so S must exceed the projected disk area 4π . The exact result does. Its integrand also increases with r , matching the increasing steepness of the paraboloid.

Question 3

Find the area of the conical surface

$$z = \sqrt{x^2 + y^2}$$

lying above the annulus $1 \leq x^2 + y^2 \leq 9$.

Tasks

1. Explain why the graph formula is valid on the annulus.
2. Evaluate the area.
3. Verify with the lateral-area formula for a conical frustum.

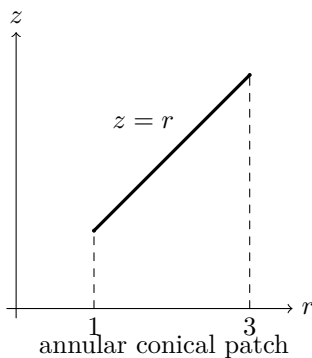
Question 3 – Solution

Strategy. The annulus avoids the cone's nondifferentiable vertex, and the cone has constant slope there.

Step 1: Gradient Write $r = \sqrt{x^2 + y^2}$. For $r > 0$,

$$f_x = \frac{x}{r}, \quad f_y = \frac{y}{r}, \quad f_x^2 + f_y^2 = 1.$$

The entire projection has $r \geq 1$, so no derivative singularity occurs.



Step 2: Area

$$S = \iint_D \sqrt{1 + 1} \, dA = \sqrt{2} \pi (3^2 - 1^2) = \boxed{8\sqrt{2} \pi}.$$

Verification The frustum has radii 1 and 3. Its slant height is $\sqrt{(3-1)^2 + (3-1)^2} = 2\sqrt{2}$, so the classical lateral area $\pi(R+r)\ell = \pi(3+1)(2\sqrt{2}) = 8\sqrt{2}\pi$ agrees.

Question 4

Use a surface-area integral to find the area of the upper hemisphere

$$z = \sqrt{9 - x^2 - y^2}.$$

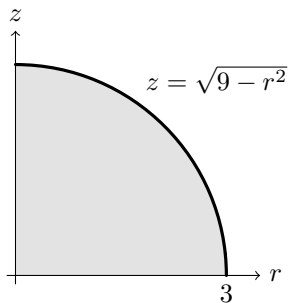
Tasks

1. Derive the surface-area factor over the open disk $x^2 + y^2 < 9$.
2. Evaluate the resulting improper integral.
3. Explain why the boundary singularity is integrable.

Question 4 – Solution

Strategy. Integrate over disks of radius $b < 3$, then take the limit as $b \rightarrow 3^-$.

Step 1: Area factor



With $r^2 = x^2 + y^2$,

$$f_x = -\frac{x}{\sqrt{9 - r^2}}, \quad f_y = -\frac{y}{\sqrt{9 - r^2}},$$

so

$$\sqrt{1 + f_x^2 + f_y^2} = \frac{3}{\sqrt{9 - r^2}}.$$

Step 2: Improper integral

$$\begin{aligned} S &= \lim_{b \rightarrow 3} \int_0^{2\pi} \int_0^b \frac{3r}{\sqrt{9 - r^2}} dr d\theta \\ &= \lim_{b \rightarrow 3} 6\pi \left(3 - \sqrt{9 - b^2} \right) = \boxed{18\pi}. \end{aligned}$$

Verification The factor diverges as $r \rightarrow 3^-$ because the sphere becomes vertical, but its antiderivative has a finite limit. The result also equals half the full sphere area $4\pi(3^2) = 36\pi$.

Question 5

Find the area of the sideways paraboloid

$$x = y^2 + z^2$$

over the disk $y^2 + z^2 \leq 1$ in the yz -plane.

Tasks

1. State the surface-area formula for a graph $x = g(y, z)$.
2. Evaluate using polar coordinates in the yz -plane.
3. Check the result against the projected disk area.

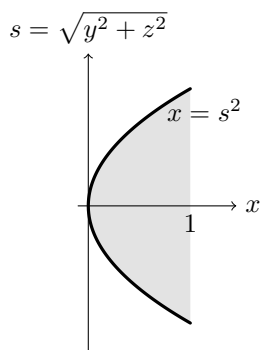
Question 5 – Solution

Strategy. Project onto the yz -plane, where both the region and squared gradient are radial.

Step 1: Projection formula For $x = g(y, z)$,

$$dS = \sqrt{1 + g_y^2 + g_z^2} dy dz.$$

Here $g = y^2 + z^2$, so with $s^2 = y^2 + z^2$ the factor is $\sqrt{1 + 4s^2}$.



Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4s^2} s ds d\theta \\ &= 2\pi \left[\frac{(1 + 4s^2)^{3/2}}{12} \right]_0^1 = \boxed{\frac{\pi}{6} (5\sqrt{5} - 1)}. \end{aligned}$$

Verification The graph projects one-to-one onto a disk of area π , and the area factor exceeds 1 except at its center. Accordingly, the computed surface area is strictly larger than π .

Question 6

Find the area of the portion of

$$z = 4 - x - y$$

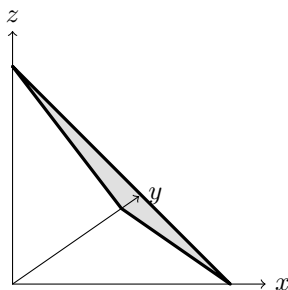
in the first octant.

Tasks

1. Determine the projection in the xy -plane.
2. Evaluate the graph surface-area integral.
3. Verify using the three intercept vertices.

Question 6 – Solution

Strategy. The coordinate-plane restrictions cut a triangular patch from a plane with constant slope.

Step 1: Region

In the first octant, $x \geq 0$, $y \geq 0$, and $z = 4 - x - y \geq 0$. Thus the projection is

$$D = \{(x, y) : 0 \leq x \leq 4, 0 \leq y \leq 4 - x\},$$

a right triangle of area 8.

Step 2: Surface integral Since $f_x = f_y = -1$,

$$S = \iint_D \sqrt{1 + (-1)^2 + (-1)^2} dA = \sqrt{3}(8) = \boxed{8\sqrt{3}}.$$

Verification From $(4, 0, 0)$, use edge vectors $\mathbf{a} = \langle -4, 4, 0 \rangle$ and $\mathbf{b} = \langle -4, 0, 4 \rangle$. Then $|\mathbf{a} \times \mathbf{b}| = 16\sqrt{3}$, so the triangle area is $\frac{1}{2}(16\sqrt{3}) = 8\sqrt{3}$.

Question 7

Find the area of the surface

$$z = xy$$

above the quarter-disk $x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$.

Tasks

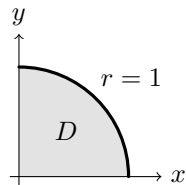
1. Show that the surface-area factor is radial even though $z = xy$ is not.
2. Evaluate the polar integral.
3. Verify the angular and radial bounds.

Question 7 – Solution

Strategy. Differentiate first: the two gradient components exchange x and y , so their squares combine to r^2 .

Step 1: Gradient and projection

$$f_x = y, \quad f_y = x, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}.$$



The quarter-disk has $0 \leq \theta \leq \pi/2$ and $0 \leq r \leq 1$.

Step 2: Evaluate

$$\begin{aligned} S &= \int_0^{\pi/2} \int_0^1 \sqrt{1 + r^2} r \, dr \, d\theta \\ &= \frac{\pi}{2} \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^1 = \boxed{\frac{\pi}{6}(2\sqrt{2} - 1)}. \end{aligned}$$

Verification The angular width is one quarter of a full turn. The factor ranges from 1 to $\sqrt{2}$, so the result must lie between the projection area $\pi/4$ and $\sqrt{2}\pi/4$; it does.

Question 8

For constants a, b, c , the plane

$$z = ax + by + c$$

lies above the unit disk $x^2 + y^2 \leq 1$. Its surface area there is 3π .

Tasks

1. Determine all possible gradient pairs (a, b) .
2. Explain the role of c .
3. Give two distinct planes satisfying the condition and verify them.

Question 8 – Solution

Strategy. A plane has a constant area-enlargement factor determined only by the magnitude of its gradient.

Step 1: Area equation Since $f_x = a$ and $f_y = b$,

$$S = \iint_D \sqrt{1 + a^2 + b^2} \, dA = \pi \sqrt{1 + a^2 + b^2}.$$

The condition $S = 3\pi$ gives

$$1 + a^2 + b^2 = 9, \quad \boxed{a^2 + b^2 = 8}.$$

Thus all possible gradients form the circle of radius $2\sqrt{2}$ in the ab -plane.

Step 2: Vertical translation The constant c does not appear in either derivative, so

$$\boxed{c \text{ is arbitrary}}.$$

Changing c translates the patch vertically without stretching it.

Step 3: Examples and verification Two choices are

$$\boxed{z = 2\sqrt{2}x} \quad \text{and} \quad \boxed{z = 2x + 2y + 5}.$$

Their gradient-square sums are respectively $8 + 0$ and $4 + 4$. Each therefore has area factor $\sqrt{1 + 8} = 3$ and area 3π above the unit disk.

Question 9

Find the area of the saddle

$$z = \frac{x^2 - y^2}{2}$$

above the disk $x^2 + y^2 \leq 3$.

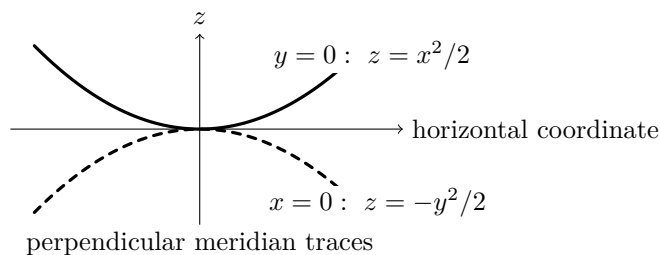
Tasks

1. Relate the two perpendicular meridian traces to the saddle shape.
2. Show that the surface-area factor is radial.
3. Evaluate the area exactly.

Question 9 – Solution

Strategy. Although the height changes sign by direction, the squared gradient depends only on distance from the origin.

Step 1: Geometry



Along $y = 0$ the trace opens upward, while along $x = 0$ it opens downward. Differentiation gives

$$f_x = x, \quad f_y = -y, \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + r^2}.$$

Step 2: Polar integral

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + r^2} r \, dr \, d\theta \\ &= 2\pi \left[\frac{(1 + r^2)^{3/2}}{3} \right]_0^{\sqrt{3}} \\ &= \frac{2\pi}{3} (8 - 1) = \boxed{\frac{14\pi}{3}}. \end{aligned}$$

Verification The projected disk has area 3π , while the area factor ranges from 1 to 2. Hence $3\pi < S < 6\pi$, and $14\pi/3$ lies in that interval.

Question 10

Let f be continuously differentiable on a region D of area 12, and suppose

$$1 \leq |\nabla f(x, y)| \leq 2 \quad \text{throughout } D.$$

Let S be the area of the graph $z = f(x, y)$ above D .

Tasks

1. Derive the sharpest bounds guaranteed for S .
2. Prove the bounds from the surface-area integral.
3. Show that both endpoints can occur.

Question 10 – Solution

Strategy. Apply the monotonicity of $\sqrt{1+t^2}$ to the pointwise gradient bounds, then integrate.

Step 1: Pointwise comparison From $1 \leq |\nabla f| \leq 2$,

$$\sqrt{2} = \sqrt{1+1^2} \leq \sqrt{1+|\nabla f|^2} \leq \sqrt{1+2^2} = \sqrt{5}.$$

Step 2: Integrate The graph area is

$$S = \iint_D \sqrt{1+|\nabla f|^2} dA.$$

Integrating the pointwise inequalities and using $\text{area}(D) = 12$ yields

$$\boxed{12\sqrt{2} \leq S \leq 12\sqrt{5}}.$$

Step 3: Sharpness The lower endpoint occurs for any plane with constant gradient magnitude 1, for example $f(x, y) = x$. The upper endpoint occurs for a plane with constant gradient magnitude 2, for example $f(x, y) = 2x$. Indeed,

$$S_{f=x} = 12\sqrt{1+1} = 12\sqrt{2}, \quad S_{f=2x} = 12\sqrt{1+4} = 12\sqrt{5}.$$

Therefore no narrower interval can be guaranteed from the supplied data.