

Question 1

Let

$$R = \{(x, y) : |x + y| \leq 2, |x - y| \leq 1\}.$$

Use $u = x + y$ and $v = x - y$ to evaluate

$$\iint_R (x + y)^2 dA.$$

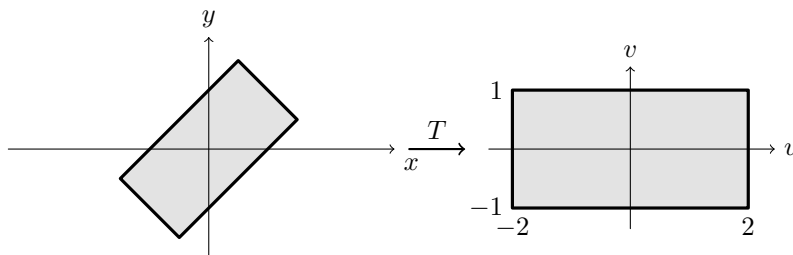
Tasks

1. Describe the image of R in the uv -plane.
2. Compute the inverse Jacobian and evaluate the integral.
3. Verify the sign and scale of the answer.

Question 1 – Solution

Strategy. The boundary expressions themselves become coordinates, turning the tilted rectangle into an axis-aligned rectangle.

Step 1: Transform the region



The inequalities become $-2 \leq u \leq 2$ and $-1 \leq v \leq 1$. Solving $u = x + y$, $v = x - y$ gives

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}.$$

Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}, \quad dA = \frac{1}{2} du dv.$$

Step 3: Evaluate

$$\begin{aligned} I &= \int_{-2}^2 \int_{-1}^1 u^2 \left(\frac{1}{2} \right) dv du \\ &= \frac{1}{2}(2) \left[\frac{u^3}{3} \right]_{-2}^2 = \boxed{\frac{16}{3}}. \end{aligned}$$

Verification The integrand is nonnegative. The transformed rectangle has area 8, scaled by $1/2$, so $\text{area}(R) = 4$; the result corresponds to average value $4/3$, which is below the maximum $u^2 = 4$.

Question 2

In the first quadrant, let R be bounded by

$$xy = 1, \quad xy = 4, \quad \frac{x}{y} = 1, \quad \frac{x}{y} = 9.$$

Use $u = xy$ and $v = x/y$ to evaluate $\iint_R (x/y) \, dA$.

Tasks

1. Show that the transformation is one-to-one on R .
2. Find the inverse Jacobian.
3. Evaluate and check the result against the integrand range.

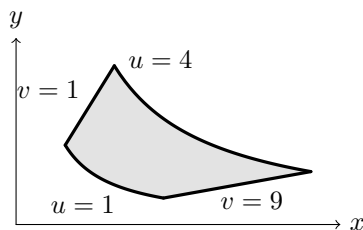
Question 2 – Solution

Strategy. Product and ratio coordinates straighten the hyperbolas and rays into a rectangle.

Step 1: Region and inverse In the first quadrant,

$$x = \sqrt{uv}, \quad y = \sqrt{\frac{u}{v}},$$

so positive (u, v) determine a unique (x, y) . The image is $1 \leq u \leq 4$, $1 \leq v \leq 9$.



Step 2: Jacobian Differentiating the inverse formulas gives

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \frac{1}{2v}.$$

Therefore the factor $x/y = v$ cancels the v in the denominator.

Step 3: Evaluate

$$I = \int_1^4 \int_1^9 v \left(\frac{1}{2v} \right) dv du = \frac{1}{2}(3)(8) = \boxed{12}.$$

Verification The region area is $\int_1^4 \int_1^9 (2v)^{-1} dv du = 3 \ln 3$. Since $1 \leq x/y \leq 9$, the bounds $3 \ln 3 \leq I \leq 27 \ln 3$ hold; $I = 12$ lies between them.

Question 3

Let

$$E = \left\{ (x, y) : \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \right\}.$$

Use $x = 3u$, $y = 2v$ and then polar coordinates in the uv -plane to evaluate

$$\iint_E \left(\frac{x^2}{9} + \frac{y^2}{4} \right) dA.$$

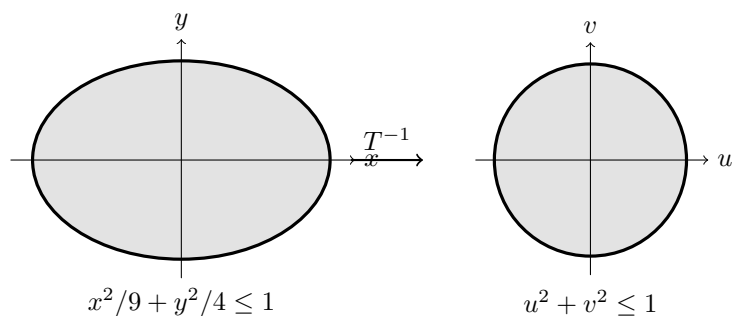
Tasks

1. Map the ellipse to a unit disk.
2. Track both Jacobian factors and evaluate.
3. Verify by comparing the integral with the ellipse area.

Question 3 – Solution

Strategy. First remove the unequal axis scales, then exploit radial symmetry on the resulting unit disk.

Step 1: Linear scaling



The map $x = 3u$, $y = 2v$ sends the unit disk D to E , with

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = 6, \quad \frac{x^2}{9} + \frac{y^2}{4} = u^2 + v^2.$$

Step 2: Polar coordinates in D Write $u = r \cos \theta$, $v = r \sin \theta$ and $du dv = r dr d\theta$. Then

$$\begin{aligned} I &= 6 \int_0^{2\pi} \int_0^1 r^2 (r dr d\theta) \\ &= 6(2\pi) \left[\frac{r^4}{4} \right]_0^1 = \boxed{3\pi}. \end{aligned}$$

Verification The ellipse area is 6π , and the transformed radial-square function has average $1/2$ on the unit disk. Thus the integral must be $(6\pi)(1/2) = 3\pi$.

Question 4

Let R be bounded by the four lines

$$x + 2y = 0, \quad x + 2y = 2, \quad 3x - y = 1, \quad 3x - y = 4.$$

Use $u = x + 2y$, $v = 3x - y$ to evaluate

$$\iint_R (x + 2y)(3x - y) \, dA.$$

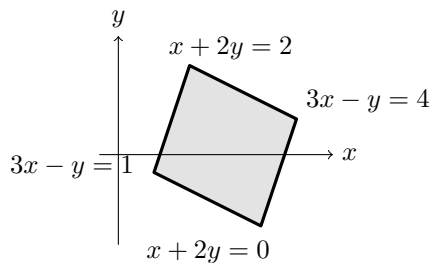
Tasks

1. Identify the transformed rectangle.
2. Compute the Jacobian and integral.
3. Verify that the transformation preserves one-to-one coverage.

Question 4 – Solution

Strategy. Each pair of parallel boundary lines becomes a pair of constant-coordinate sides.

Step 1: Geometry



The image is the rectangle $0 \leq u \leq 2$, $1 \leq v \leq 4$. The coefficient matrix has determinant

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7.$$

Since this determinant is nonzero, the linear map is globally one-to-one.

Step 2: Change variables We have $dA = \frac{1}{7} du dv$ and the integrand is uv . Hence

$$\begin{aligned} I &= \frac{1}{7} \int_0^2 \int_1^4 uv \, dv \, du \\ &= \frac{1}{7} \left[\frac{u^2}{2} \right]_0^2 \left[\frac{v^2}{2} \right]_1^4 = \boxed{\frac{15}{7}}. \end{aligned}$$

Verification On the transformed rectangle, $u \geq 0$ and $v > 0$, so the positive answer has the required sign. The constant area scale $1/7$ also matches the inverse determinant.

Question 5

Let

$$R = \{(x, y) : 1 \leq x \leq 3, x \leq y \leq 2x\}.$$

Use $u = x$ and $v = y/x$ to evaluate

$$\iint_R \frac{1}{x} dA.$$

Tasks

1. Find the inverse map and transformed bounds.
2. Compute the Jacobian and evaluate.
3. Verify directly in Cartesian coordinates.

Question 5 – Solution

Strategy. The ratio $v = y/x$ turns the two sloping sides into constant-coordinate boundaries.

Step 1: Inverse and bounds Since $x > 0$ on R ,

$$x = u, \quad y = uv.$$

The region becomes the rectangle

$$1 \leq u \leq 3, \quad 1 \leq v \leq 2.$$

The inverse map is one-to-one because $u = x$ and $v = y/x$ recover each point uniquely.

Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u.$$

Thus

$$I = \int_1^3 \int_1^2 \frac{1}{u}(u) \, dv \, du = \int_1^3 \int_1^2 1 \, dv \, du = \boxed{2}.$$

Verification Direct Cartesian integration gives

$$\int_1^3 \int_x^{2x} \frac{1}{x} \, dy \, dx = \int_1^3 \frac{2x - x}{x} \, dx = 2.$$

The two methods agree exactly, including the variable area scale u .

Question 6

Consider the map

$$T(u, v) = (u^2, v), \quad -1 \leq u \leq 1, \quad 0 \leq v \leq 1.$$

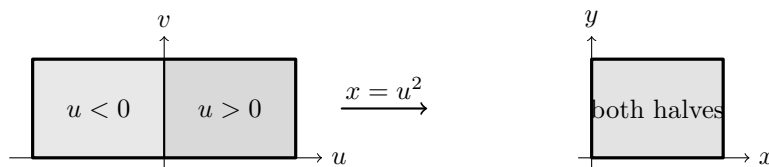
Tasks

1. Describe the image and determine whether T is one-to-one.
2. Explain why integrating $|J_T|$ over the full source gives the wrong image area.
3. Repair the change of variables calculation.

Question 6 – Solution

Strategy. Track multiplicity: the two halves of the source fold onto the same target rectangle.

Step 1: Image and folding



The image is $0 \leq x \leq 1$, $0 \leq y \leq 1$. Except along $u = 0$, the points (u, v) and $(-u, v)$ have the same image, so T is two-to-one.

Step 2: Diagnose the overcount Since

$$J_T = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & 0 \\ 0 & 1 \end{vmatrix} = 2u,$$

integrating its absolute value over the full source gives

$$\int_{-1}^1 \int_0^1 |2u| \, dv \, du = 2,$$

twice the actual image area 1. The absolute value corrects orientation, not multiplicity.

Step 3: Repair Restrict to the one-to-one half $0 \leq u \leq 1$:

$$\text{area}(T(S)) = \int_0^1 \int_0^1 2u \, dv \, du = \boxed{1}.$$

Using $-1 \leq u \leq 0$ alone gives the same result after taking $|J_T|$.

Question 7

Let

$$R = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq x^2 + 3\}.$$

Use $u = x$ and $v = y - x^2$ to evaluate

$$\iint_R (y - x^2) dA.$$

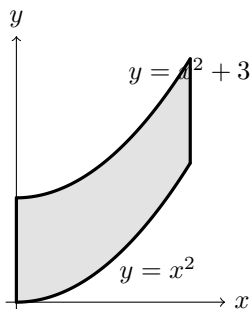
Tasks

1. Show how the change of variables straightens the parabolic strip.
2. Compute the Jacobian and integral.
3. Verify directly from the vertical thickness.

Question 7 – Solution

Strategy. Measure vertical displacement from the lower parabola; the curved strip then becomes a rectangle.

Step 1: Straighten the region



The inverse map is

$$x = u, \quad y = u^2 + v,$$

and the bounds become $0 \leq u \leq 2$, $0 \leq v \leq 3$.

Step 2: Jacobian and integral

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 2u & 1 \end{vmatrix} = 1.$$

Because $y - x^2 = v$,

$$I = \int_0^2 \int_0^3 v \, dv \, du = (2) \left[\frac{v^2}{2} \right]_0^3 = \boxed{9}.$$

Verification For each fixed x , the displacement $t = y - x^2$ ranges from 0 to 3. Thus direct integration gives $\int_0^2 \int_0^3 t \, dt \, dx = 9$. The unit Jacobian reflects a vertical shear with no area scaling.

Question 8

The unit cube $0 \leq u, v, w \leq 1$ is mapped into xyz -space by

$$x = u + v, \quad y = v + w, \quad z = w + u.$$

Evaluate $\iiint_E (x + y + z) dV$, where E is the image parallelepiped.

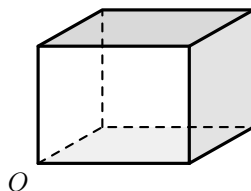
Tasks

1. Compute the three-dimensional Jacobian and establish invertibility.
2. Evaluate the transformed integral.
3. Verify using the image volume and centroid.

Question 8 – Solution

Strategy. A nonsingular linear map sends the unit cube to a parallelepiped with constant volume scale.

Step 1: Jacobian and geometry



$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2.$$

Because $J \neq 0$, the linear map is invertible and covers its image once.

Step 2: Transform the integrand We have $x + y + z = 2(u + v + w)$ and $dV = 2 du dv dw$. Therefore

$$\begin{aligned} I &= \int_0^1 \int_0^1 \int_0^1 4(u + v + w) dw dv du \\ &= 4 \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = \boxed{6}. \end{aligned}$$

Verification The image volume is $|J|$ times the cube volume, hence 2. The cube centroid $(1/2, 1/2, 1/2)$ maps to $(1, 1, 1)$, where $x + y + z = 3$. A linear function averages to its centroid value, so $I = (2)(3) = 6$.

Question 9

For $a, b > 0$, let

$$E_{a,b} = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Suppose

$$\text{area}(E_{a,b}) = 6\pi, \quad \iint_{E_{a,b}} x^2 \, dA = \frac{27\pi}{2}.$$

Tasks

1. Use $x = au$, $y = bv$ to translate both measurements.
2. Recover a and b .
3. Verify both original conditions.

Question 9 – Solution

Strategy. Scale the ellipse to the unit disk; area determines ab , while the x^2 moment determines one additional power of a^2 .

Step 1: Area condition The map $x = au$, $y = bv$ has $dA = ab \, du \, dv$. Since the unit disk has area π ,

$$\pi ab = 6\pi, \quad \text{so} \quad ab = 6.$$

Step 2: Moment condition On the unit disk,

$$\iint_D u^2 \, du \, dv = \frac{\pi}{4}$$

by symmetry or a polar calculation. Therefore

$$\iint_{E_{a,b}} x^2 \, dA = a^2(ab) \frac{\pi}{4} = \frac{3\pi a^2}{2}.$$

Equating this with $27\pi/2$ gives $a^2 = 9$. Positivity then yields

$$\boxed{a = 3}, \quad \boxed{b = 2}.$$

Verification The recovered ellipse has area $\pi(3)(2) = 6\pi$. Its moment is $a^2(ab)\pi/4 = (9)(6)\pi/4 = 27\pi/2$, so both measurements are satisfied without sign ambiguity.

Question 10

Let

$$R = \{(x, y) : x \geq 0, y \geq 0, 1 \leq x + y \leq 3\}.$$

Use

$$u = x + y, \quad v = \frac{x}{x + y}$$

to evaluate $\iint_R (x + y)^{-1} dA$.

Tasks

1. Derive the inverse map and transformed rectangle.
2. Compute the Jacobian and integral.
3. Explain geometrically what the coordinate v measures.

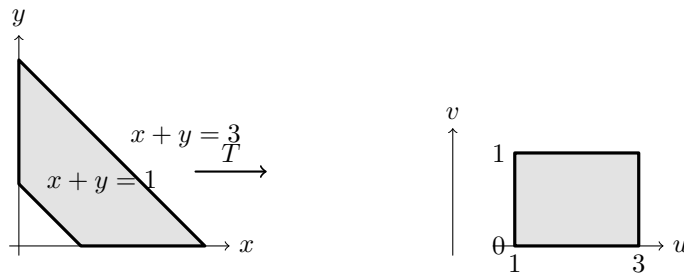
Question 10 – Solution

Strategy. Use total distance along the coordinate-sum direction and the fraction allocated to x .

Step 1: Inverse and geometry Since $u = x + y > 0$,

$$x = uv, \quad y = u(1 - v).$$

Nonnegativity gives $0 \leq v \leq 1$, while the two diagonal boundaries give $1 \leq u \leq 3$.



Step 2: Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} v & u \\ 1 - v & -u \end{vmatrix} = -u, \quad dA = u \, du \, dv.$$

Hence the integrand $1/(x + y) = 1/u$ cancels the area scale:

$$I = \int_1^3 \int_0^1 \frac{1}{u}(u) \, dv \, du = \boxed{2}.$$

Step 3: Interpretation and check On a segment $x + y = u$, the quantity $v = x/u$ is the fraction of the total coordinate sum assigned to x ; it runs once from the y -axis to the x -axis. Thus the inverse is one-to-one in the interior, and $u > 0$ confirms the absolute Jacobian used above.