

Question 1

Let E be the part of the spherical shell

$$1 \leq \sqrt{x^2 + y^2 + z^2} \leq 3$$

in the first octant. Evaluate

$$\iiint_E \frac{1}{\sqrt{x^2 + y^2 + z^2}} dV$$

using spherical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the integral.
3. Check the result using the integrand range and region volume.

Question 1 – Solution

Strategy. A first-octant spherical shell has constant bounds in all three spherical variables, and the radial integrand cancels one power of the Jacobian.

Step 1: Convert Using

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi,$$

the bounds are

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \frac{\pi}{2}, \quad 1 \leq \rho \leq 3,$$

and $dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$.

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^{\pi/2} \int_1^3 \frac{1}{\rho} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{2}\right) (1) \left[\frac{\rho^2}{2}\right]_1^3 = \boxed{2\pi}. \end{aligned}$$

Verification The shell-octant volume is $\frac{1}{8} \frac{4\pi}{3} (3^3 - 1^3) = 13\pi/3$. Hence the average integrand is $I/V = 6/13$, which lies between its endpoint values $1/3$ and 1 .

Question 2

Find the volume inside the sphere $x^2 + y^2 + z^2 \leq 16$ and above the cone

$$z = \sqrt{3}\sqrt{x^2 + y^2}.$$

Tasks

1. Convert the cone to a constant ϕ -boundary.
2. Set up and evaluate the spherical integral.
3. Compare with the volume of the containing ball.

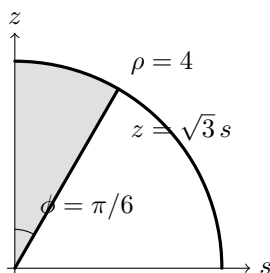
Question 2 – Solution

Strategy. In a meridian plane, the cone fixes the polar angle while the sphere fixes the radial endpoint.

Step 1: Angular boundary Since $z = \rho \cos \phi$ and $\sqrt{x^2 + y^2} = \rho \sin \phi$,

$$\rho \cos \phi = \sqrt{3} \rho \sin \phi \implies \tan \phi = \frac{1}{\sqrt{3}}.$$

The region above the cone therefore has $0 \leq \phi \leq \pi/6$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/6} \int_0^4 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 2\pi \left(1 - \frac{\sqrt{3}}{2}\right) \frac{64}{3} = \boxed{\frac{64\pi}{3}(2 - \sqrt{3})}. \end{aligned}$$

Verification The angular factor $1 - \sqrt{3}/2$ is between 0 and 1, so this positive sector is smaller than the upper half-ball and therefore smaller than the entire ball of volume $256\pi/3$.

Question 3

The upper half of the spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$.

Tasks

1. Find its mass in spherical coordinates.
2. Find its average density.
3. Verify the average against the density range.

Question 3 – Solution

Strategy. The hemisphere contributes the full azimuthal interval but only $0 \leq \phi \leq \pi/2$; the radial density adds one power of ρ .

Step 1: Mass

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^{\pi/2} \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi)(1) \left[\frac{\rho^4}{4} \right]_1^2 = \boxed{\frac{15\pi}{2}}. \end{aligned}$$

Step 2: Average density The region volume is half the difference of two ball volumes:

$$V = \frac{1}{2} \frac{4\pi}{3} (2^3 - 1^3) = \frac{14\pi}{3}.$$

Therefore

$$\boxed{\delta_{\text{avg}} = \frac{M}{V} = \frac{45}{28}}.$$

Verification Since $1 \leq \delta = \rho \leq 2$ throughout the shell, the value $45/28 \approx 1.61$ is admissible. The larger outer spherical layers correctly pull it above the midpoint $3/2$.

Question 4

Let E be the upper half of the ball $x^2 + y^2 + z^2 \leq 9$.

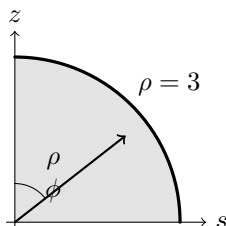
Tasks

1. Use spherical coordinates to compute $\iiint_E z \, dV$.
2. Find the average height z_{avg} .
3. Check its geometric range.

Question 4 – Solution

Strategy. Write $z = \rho \cos \phi$ and use the upper-hemisphere bounds $0 \leq \phi \leq \pi/2$.

Step 1: Geometry and integral



$$\begin{aligned}
 I &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= (2\pi) \left[\frac{\sin^2 \phi}{2} \right]_0^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^3 = \boxed{\frac{81\pi}{4}}.
 \end{aligned}$$

Step 2: Average height The hemisphere volume is $V = \frac{1}{2} \frac{4\pi}{3} (3^3) = 18\pi$. Thus

$$z_{\text{avg}} = \frac{I}{V} = \frac{9}{8}.$$

Verification Every point has $0 \leq z \leq 3$, and $9/8$ lies in that interval. The value is below the mid-height $3/2$ because cross-sectional area is greatest near $z = 0$.

Question 5

Inside a ball of radius R , consider the cone sector

$$0 \leq \rho \leq R, \quad 0 \leq \phi \leq \alpha, \quad 0 \leq \theta \leq 2\pi,$$

where $0 < \alpha < \pi/2$. Its volume is one-third of the upper half-ball.

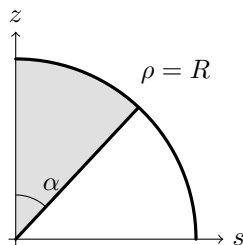
Tasks

1. Determine α .
2. Write the boundary cone in Cartesian coordinates.
3. Verify the stated volume ratio.

Question 5 – Solution

Strategy. The radial and azimuthal factors cancel from the ratio, leaving only the polar-angle factor $1 - \cos \alpha$.

Step 1: Sector volume



$$V_{\text{sector}} = \int_0^{2\pi} \int_0^\alpha \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{2\pi R^3}{3} (1 - \cos \alpha).$$

The upper half-ball has volume $2\pi R^3/3$. Hence

$$1 - \cos \alpha = \frac{1}{3}, \quad \boxed{\alpha = \arccos\left(\frac{2}{3}\right)}.$$

Step 2: Cartesian cone On $\phi = \alpha$, $z = s \cot \alpha$, where $s = \sqrt{x^2 + y^2}$. Since $\sin \alpha = \sqrt{5}/3$, we have $\cot \alpha = 2/\sqrt{5}$. Thus

$$\boxed{z = \frac{2}{\sqrt{5}} \sqrt{x^2 + y^2}}.$$

Verification Substituting $\cos \alpha = 2/3$ makes the angular factor $1 - \cos \alpha = 1/3$, so the sector volume is exactly one-third of the upper half-ball volume.

Question 6

Use spherical coordinates to find the volume inside

$$x^2 + y^2 + z^2 \leq 2z.$$

Tasks

1. Convert the boundary to a radial spherical bound.
2. Evaluate the volume.
3. Identify the Cartesian center and radius as a check.

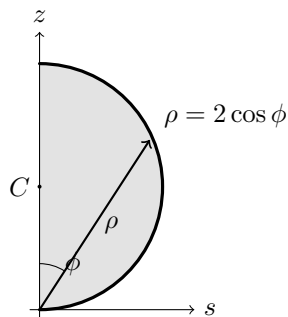
Question 6 – Solution

Strategy. The shifted sphere becomes $\rho \leq 2 \cos \phi$, so its nonconstant radial bound encodes the displacement from the origin.

Step 1: Bounds Substitution gives

$$\rho^2 \leq 2\rho \cos \phi.$$

For $\rho \geq 0$, the nontrivial radial interval is $0 \leq \rho \leq 2 \cos \phi$, which requires $0 \leq \phi \leq \pi/2$. Also $0 \leq \theta \leq 2\pi$.



Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \frac{16\pi}{3} \int_0^{\pi/2} \cos^3 \phi \sin \phi \, d\phi = \boxed{\frac{4\pi}{3}}. \end{aligned}$$

Verification Completing the square gives $x^2 + y^2 + (z - 1)^2 \leq 1$, a sphere of radius 1 centered at $(0, 0, 1)$. Its ordinary volume is $4\pi/3$, confirming the integral.

Question 7

Let

$$E = \{(x, y, z) : 2 \leq \sqrt{x^2 + y^2 + z^2} \leq 3, x \geq 0, y \geq x, z \leq 0\}.$$

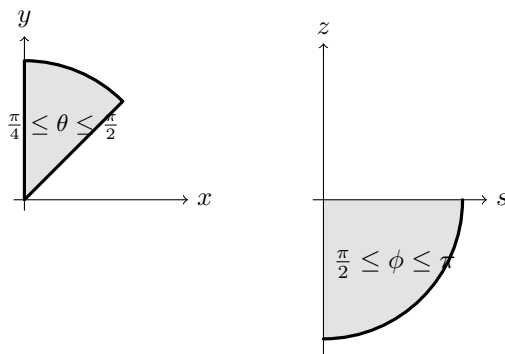
Tasks

1. Determine exact spherical bounds for E .
2. Find its volume.
3. Verify the angular fractions geometrically.

Question 7 – Solution

Strategy. Resolve the horizontal inequalities into a θ -wedge and the vertical inequality into a ϕ -hemisphere.

Step 1: Angular bounds The conditions $x \geq 0$ and $y \geq x$ select $\pi/4 \leq \theta \leq \pi/2$. The condition $z \leq 0$ selects $\pi/2 \leq \phi \leq \pi$, while the shell gives $2 \leq \rho \leq 3$.



Step 2: Volume

$$\begin{aligned} V &= \int_{\pi/4}^{\pi/2} \int_{\pi/2}^{\pi} \int_2^3 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= \left(\frac{\pi}{4}\right) (1) \left(\frac{27-8}{3}\right) = \boxed{\frac{19\pi}{12}}. \end{aligned}$$

Verification The azimuthal wedge is $(\pi/4)/(2\pi) = 1/8$ of a full turn, and $z \leq 0$ supplies half the polar solid angle. Thus E occupies $1/16$ of the full shell; indeed $\frac{1}{16} \frac{4\pi}{3} (27-8) = 19\pi/12$.

Question 8

A full spherical shell $1 \leq \rho \leq 2$ has density $\delta = \rho$. Find its moment of inertia about the z -axis.

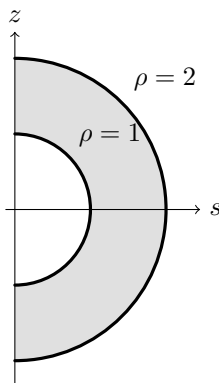
Tasks

1. Express the squared distance to the z -axis in spherical coordinates.
2. Compute the mass and I_z .
3. Find and check the radius of gyration $k_z = \sqrt{I_z/M}$.

Question 8 – Solution

Strategy. The distance to the z -axis is $\rho \sin \phi$; combine its square with the radial density and spherical Jacobian.

Step 1: Region and mass



$$M = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = (2\pi)(2) \frac{15}{4} = \boxed{15\pi}.$$

Step 2: Moment of inertia

$$\begin{aligned} I_z &= \iiint_E (\rho \sin \phi)^2 \delta \, dV \\ &= \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^5 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\ &= (2\pi) \left(\frac{4}{3} \right) \left(\frac{2^6 - 1}{6} \right) = \boxed{28\pi}. \end{aligned}$$

Step 3: Radius of gyration

$$\boxed{k_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{28}{15}}}.$$

It lies between the minimum and maximum distances 0 and 2 from the axis, as required.

Question 9

Explain why spherical coordinates use

$$dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,$$

then evaluate $\iiint_E \rho^{-2} \, dV$ on the shell $1 \leq \rho \leq 3$.

Tasks

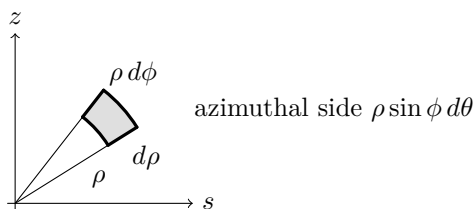
1. Derive the three local scale factors geometrically.
2. Evaluate the integral over the full shell.
3. Check the dimensional scaling and behavior near the origin.

Question 9 – Solution

Strategy. Approximate a small spherical cell by a box with one radial side and two angular arc-length sides.

Step 1: Volume element The local side lengths are

$$d\rho, \quad \rho d\phi, \quad \rho \sin \phi d\theta.$$



Their product is

$$(d\rho)(\rho d\phi)(\rho \sin \phi d\theta) = \rho^2 \sin \phi d\rho d\phi d\theta.$$

Step 2: Apply

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\pi \int_1^3 \rho^{-2} \rho^2 \sin \phi d\rho d\phi d\theta \\ &= (2\pi)(2)(3-1) = \boxed{8\pi}. \end{aligned}$$

Verification The shell excludes $\rho = 0$, so the integrand is bounded. Its units are length^{-2} ; multiplying by volume produces length, matching the radial factor $3 - 1$ in the result. Even if the inner radius tends to 0, the radial integral remains finite: $4\pi \int_0^3 d\rho = 12\pi$. Thus the singularity is integrable.

Question 10

For $a > 0$, a uniform spherical shell occupies

$$a \leq \rho \leq 2a.$$

Its average distance from the origin is $45/7$.

Tasks

1. Derive the shell volume as a function of a .
2. Derive the average value of ρ .
3. Recover a , the outer radius, and the shell volume.

Question 10 – Solution

Strategy. Divide the first radial moment by volume; the common angular factors cancel, leaving a linear function of a .

Step 1: Volume

$$\begin{aligned} V(a) &= \int_0^{2\pi} \int_0^\pi \int_a^{2a} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\ &= 4\pi \left[\frac{\rho^3}{3} \right]_a^{2a} = \frac{28\pi a^3}{3}. \end{aligned}$$

Step 2: Average radius The first radial moment is

$$M_\rho(a) = 4\pi \int_a^{2a} \rho^3 \, d\rho = 15\pi a^4.$$

Hence

$$\rho_{\text{avg}} = \frac{M_\rho(a)}{V(a)} = \frac{45a}{28}.$$

Step 3: Recover the shell Setting $45a/28 = 45/7$ gives $\boxed{a = 4}$. Therefore

$$\boxed{\text{outer radius} = 8}, \quad \boxed{V = \frac{1792\pi}{3}}.$$

Verification The average $45/7 \approx 6.43$ lies strictly between the recovered radii 4 and 8. Substituting $a = 4$ into $28\pi a^3/3$ gives the stated volume.