

Question 1

Let

$$E = \{(x, y, z) : x^2 + y^2 \leq 4, x \geq 0, y \geq 0, -1 \leq z \leq 2\}.$$

Evaluate $\iiint_E (x^2 + y^2) dV$ using cylindrical coordinates.

Tasks

1. Convert the region, integrand, and volume element.
2. Evaluate the cylindrical integral.
3. Verify the result using an average value.

Question 1 – Solution

Strategy. Treat the solid as a quarter-cylinder; radial symmetry makes the integrand r^2 .

Step 1: Convert Cylindrical coordinates give

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2, \quad dV = r \, dz \, dr \, d\theta.$$

The first-quadrant quarter-cylinder has bounds

$$0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2, \quad -1 \leq z \leq 2.$$

Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^2 \int_{-1}^2 r^2 (r \, dz \, dr \, d\theta) \\ &= \left(\frac{\pi}{2}\right) (3) \left[\frac{r^4}{4}\right]_0^2 = \boxed{6\pi}. \end{aligned}$$

Verification The volume is $(\pi 2^2/4)(3) = 3\pi$. On a radius-2 disk, the average of r^2 is $2^2/2 = 2$; restricting to one quadrant does not change that radial average. Thus $I = (3\pi)(2) = 6\pi$.

Question 2

Find the volume inside the paraboloid

$$z = 9 - x^2 - y^2$$

and above the xy -plane.

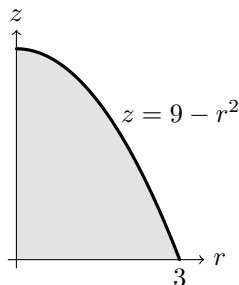
Tasks

1. Determine the cylindrical bounds from the surface intersection.
2. Evaluate the volume.
3. Check the answer against a containing cylinder.

Question 2 – Solution

Strategy. In cylindrical coordinates the paraboloid becomes $z = 9 - r^2$; its intersection with $z = 0$ determines the radial endpoint.

Step 1: Geometry Setting $z = 0$ gives $r = 3$. A meridian half-plane shows the height above each radius.



Thus $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 3$, and $0 \leq z \leq 9 - r^2$.

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^3 (9r - r^3) \, dr = 2\pi \left[\frac{9}{2}r^2 - \frac{1}{4}r^4 \right]_0^3 = \boxed{\frac{81\pi}{2}}. \end{aligned}$$

Verification The solid lies inside the cylinder of radius 3 and height 9, whose volume is 81π . The computed volume is positive and exactly half that containing-cylinder volume.

Question 3

Let E lie above the cone $z = \sqrt{x^2 + y^2}$ and below the paraboloid $z = 2 - x^2 - y^2$.

Tasks

1. Find the curve where the two surfaces meet.
2. Set up and evaluate $\iiint_E 1 \, dV$ in cylindrical coordinates.
3. Verify that the vertical bounds are correctly ordered.

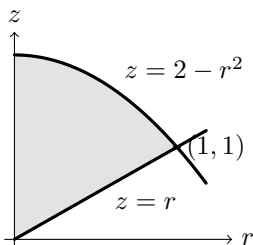
Question 3 – Solution

Strategy. Convert both surfaces to functions of r , solve for their positive intersection radius, and integrate top minus bottom.

Step 1: Intersection The surfaces are $z = r$ and $z = 2 - r^2$. Their intersection satisfies

$$r = 2 - r^2 \iff (r - 1)(r + 2) = 0.$$

Since $r \geq 0$, the boundary circle is $r = 1, z = 1$.



Step 2: Integral

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 \int_r^{2-r^2} r \, dz \, dr \, d\theta \\ &= 2\pi \int_0^1 (2 - r^2 - r)r \, dr = 2\pi \left(1 - \frac{1}{4} - \frac{1}{3} \right) = \boxed{\frac{5\pi}{6}}. \end{aligned}$$

Verification For $0 \leq r \leq 1$, $2 - r^2 - r = (1 - r)(r + 2) \geq 0$, so the upper and lower z -bounds never reverse.

Question 4

Let

$$E = \{(x, y, z) : x^2 + y^2 \leq 4x, 0 \leq z \leq \sqrt{x^2 + y^2}\}.$$

Find the volume of E using cylindrical coordinates.

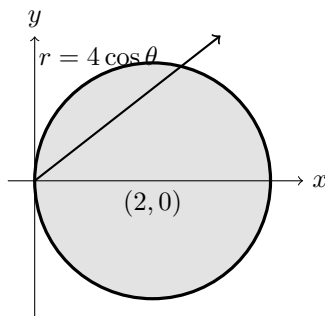
Tasks

1. Convert the offset cylindrical boundary and determine a nonduplicating angular interval.
2. Evaluate the volume.
3. Check the radial bounds at the endpoint and central rays.

Question 4 – Solution

Strategy. Convert the footprint to $r \leq 4 \cos \theta$ and use the conical height $z = r$.

Step 1: Footprint From $r^2 \leq 4r \cos \theta$, nonnegative radii satisfy $0 \leq r \leq 4 \cos \theta$. This requires $-\pi/2 \leq \theta \leq \pi/2$ and describes the disk centered at $(2, 0)$ with radius 2.



Step 2: Evaluate

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^{4 \cos \theta} \int_0^r r \, dz \, dr \, d\theta \\ &= \frac{64}{3} \int_{-\pi/2}^{\pi/2} \cos^3 \theta \, d\theta = \frac{64}{3} \left(\frac{4}{3} \right) = \boxed{\frac{256}{9}}. \end{aligned}$$

Verification At $\theta = \pm\pi/2$ the radial interval collapses to $r = 0$; at $\theta = 0$ it reaches $r = 4$. These are exactly the leftmost and rightmost points of the footprint, so no rays are omitted or duplicated.

Question 5

A solid annular cylinder is described by

$$1 \leq x^2 + y^2 \leq 4, \quad 0 \leq z \leq 3,$$

and has density $\rho(x, y, z) = \sqrt{x^2 + y^2} + z$.

Tasks

1. Find its mass using cylindrical coordinates.
2. Find its average density.
3. Compare the average with the density range.

Question 5 – Solution

Strategy. The annular cylinder becomes a rectangular box in (r, θ, z) , while the density becomes $r + z$.

Step 1: Mass The bounds are $0 \leq \theta \leq 2\pi$, $1 \leq r \leq 2$, and $0 \leq z \leq 3$. Therefore

$$\begin{aligned} M &= \int_0^{2\pi} \int_1^2 \int_0^3 (r + z)r \, dz \, dr \, d\theta \\ &= 2\pi \int_1^2 \left(3r^2 + \frac{9}{2}r \right) dr = 2\pi \left[r^3 + \frac{9}{4}r^2 \right]_1^2 = \boxed{\frac{55\pi}{2}}. \end{aligned}$$

Step 2: Average density The volume is $\pi(2^2 - 1^2)(3) = 9\pi$. Hence

$$\boxed{\rho_{\text{avg}} = \frac{M}{V} = \frac{55}{18}}.$$

Verification On the solid, $1 \leq r + z \leq 5$, and $55/18 \approx 3.06$ lies strictly in that range. The units of M/V are density units, as required.

Question 6

The cone

$$0 \leq z \leq 3 - \sqrt{x^2 + y^2}$$

has constant density $\rho = 2$.

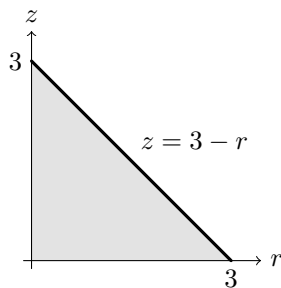
Tasks

1. Find its mass.
2. Find its moment of inertia I_z about the z -axis.
3. Find its radius of gyration $k_z = \sqrt{I_z/M}$.

Question 6 – Solution

Strategy. Use $x^2 + y^2 = r^2$ for the squared distance to the axis and $dV = r \, dz \, dr \, d\theta$.

Step 1: Region The meridian triangle gives $0 \leq r \leq 3$ and $0 \leq z \leq 3 - r$.



Step 2: Mass

$$M = \int_0^{2\pi} \int_0^3 \int_0^{3-r} 2r \, dz \, dr \, d\theta = 4\pi \int_0^3 r(3-r) \, dr = \boxed{18\pi}.$$

Step 3: Moment and radius of gyration

$$I_z = \iiint_E (x^2 + y^2) \rho \, dV = 4\pi \int_0^3 r^3(3-r) \, dr = \boxed{\frac{243\pi}{5}},$$
$$k_z = \sqrt{\frac{I_z}{M}} = \boxed{\sqrt{\frac{27}{10}}}.$$

Verification $k_z \approx 1.64$ lies between the minimum and maximum distances 0 and 3 from the axis. Also $M = 2 \cdot \frac{1}{3}\pi(3^2)(3) = 18\pi$ agrees with the cone-volume formula.

Question 7

Let E be the ball $x^2 + y^2 + z^2 \leq 4$. Without using spherical coordinates, evaluate

$$\iiint_E (x^2 + y^2) \, dV.$$

Tasks

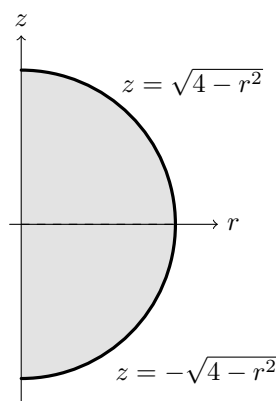
1. Express the ball with cylindrical bounds.
2. Evaluate the integral exactly.
3. Check positivity and magnitude.

Question 7 – Solution

Strategy. At fixed cylindrical radius r , the sphere supplies symmetric upper and lower z -bounds.

Step 1: Bounds Since $r^2 + z^2 \leq 4$,

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 2, \quad -\sqrt{4-r^2} \leq z \leq \sqrt{4-r^2}.$$



Step 2: Evaluate

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2 (r \, dz \, dr \, d\theta) \\ &= 4\pi \int_0^2 r^3 \sqrt{4-r^2} \, dr. \end{aligned}$$

With $u = 4 - r^2$, $r^3 dr = -(4 - u)du/2$, so

$$\int_0^2 r^3 \sqrt{4-r^2} \, dr = \frac{1}{2} \int_0^4 (4 - u) u^{1/2} du = \frac{64}{15}.$$

Thus $\boxed{I = 256\pi/15}$.

Verification The integrand is nonnegative and at most 4. Since the ball volume is $32\pi/3$, the required bound $0 < I < 128\pi/3$ holds.

Question 8

Let E be the solid between the paraboloid $z = x^2 + y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

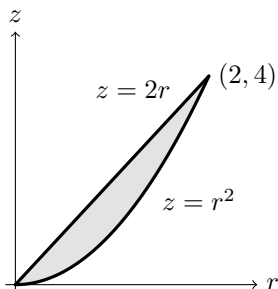
Tasks

1. Determine the cylindrical bounds.
2. Compute $\iiint_E z \, dV$.
3. Find the average value of z on E .

Question 8 – Solution

Strategy. Find the nonzero intersection radius, then integrate z between the radial graphs.

Step 1: Geometry The surfaces meet where $r^2 = 2r$, so $r = 0$ or $r = 2$. On $0 \leq r \leq 2$, the cone lies above the paraboloid.



Thus $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$, and $r^2 \leq z \leq 2r$.

Step 2: Integral of z

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^2 \int_{r^2}^{2r} zr \, dz \, dr \, d\theta \\ &= 2\pi \int_0^2 \left(2r^3 - \frac{1}{2}r^5 \right) dr = \boxed{\frac{16\pi}{3}}. \end{aligned}$$

Step 3: Average The volume is

$$V = 2\pi \int_0^2 (2r - r^2)r \, dr = \frac{8\pi}{3}.$$

Therefore $\boxed{z_{\text{avg}} = I/V = 2}$. This lies between the solid's minimum and maximum heights, 0 and 4.

Question 9

Explain why cylindrical coordinates use

$$dV = r \, dz \, dr \, d\theta,$$

then apply the formula to evaluate $\iiint_E (x^2 + y^2) \, dV$, where

$$1 \leq r \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{3}, \quad 0 \leq z \leq 4.$$

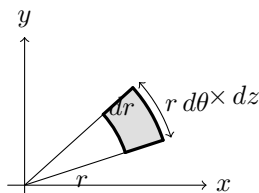
Tasks

1. Derive the scale factor r geometrically.
2. Evaluate the integral.
3. Verify the dimensional scaling.

Question 9 – Solution

Strategy. Approximate a small cylindrical cell by a rectangular box whose angular side is an arc of length $r d\theta$.

Step 1: Jacobian factor A small cell has radial length dr , vertical length dz , and circumferential length approximately $r d\theta$.



Consequently,

$$dV \approx (dr)(r d\theta)(dz) = r dz dr d\theta,$$

and the approximation becomes exact in the integral limit.

Step 2: Apply Since $x^2 + y^2 = r^2$,

$$\begin{aligned} I &= \int_0^{\pi/3} \int_1^2 \int_0^4 r^2 (r dz dr d\theta) \\ &= \left(\frac{\pi}{3}\right) (4) \left[\frac{r^4}{4}\right]_1^2 = \boxed{5\pi}. \end{aligned}$$

Verification The factors scale as angular width times height times radius to the fourth power. That produces length^5 , the correct units for integrating a squared distance over volume.

Question 10

For $a > 0$, let E_a be the solid

$$0 \leq z \leq a^2, \quad z \geq x^2 + y^2.$$

Its average z -coordinate is 6.

Tasks

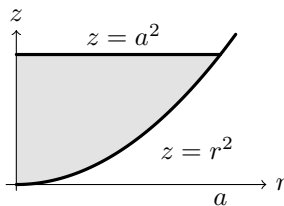
1. Express E_a in cylindrical coordinates.
2. Derive its volume and average z -coordinate as functions of a .
3. Recover a and the corresponding volume.

Question 10 – Solution

Strategy. The horizontal plane meets the paraboloid at $r = a$; compute both volume and first z -moment before imposing the average.

Step 1: Bounds The solid lies above $z = r^2$ and below $z = a^2$:

$$0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq a, \quad r^2 \leq z \leq a^2.$$



Step 2: Volume and moment

$$V(a) = 2\pi \int_0^a (a^2 - r^2)r \, dr = \frac{\pi a^4}{2},$$
$$M_z(a) = 2\pi \int_0^a \int_{r^2}^{a^2} zr \, dz \, dr = \frac{\pi a^6}{3}.$$

Therefore

$$z_{\text{avg}} = \frac{M_z(a)}{V(a)} = \frac{2a^2}{3}.$$

Step 3: Recover the parameter The condition $2a^2/3 = 6$ gives $a^2 = 9$. Since $a > 0$,

$$\boxed{a = 3}, \quad \boxed{V(3) = \frac{81\pi}{2}}.$$

Verification For $a = 3$, the solid has $0 \leq z \leq 9$, and the average 6 lies inside that range. Substitution returns $2(3^2)/3 = 6$ exactly.