

Question 1

Evaluate $\iiint_B (x + 2y + 3z) \, dV$ on $B = [0, 1] \times [0, 2] \times [-1, 1]$.

Tasks

1. Write a Cartesian iterated integral.
2. Evaluate it.
3. Verify by average values.

Question 1 – Solution

Strategy. Integrate termwise over the box.

Step 1: Set up $I = \int_0^1 \int_0^2 \int_{-1}^1 (x + 2y + 3z) dz dy dx$.

Step 2: Evaluate The z term is odd; the other terms give $2x + 4y$. Thus $I = \int_0^1 \int_0^2 (2x + 4y) dy dx = \int_0^1 (4x + 8) dx = \boxed{10}$.

Verification The box volume is 4 and its coordinate averages are $(1/2, 1, 0)$, so the integrand average is $5/2$ and $4(5/2) = 10$.

Question 2

Let $E = \{0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 2 - x - y\}$.

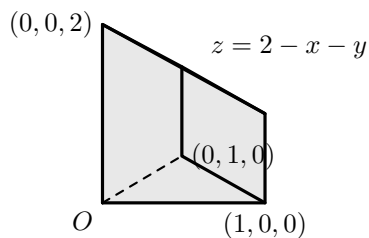
Tasks

1. Interpret the bounds.
2. Evaluate $\iiint_E 1 \, dV$.
3. Check positivity of every bound.

Question 2 – Solution

Strategy. Integrate vertical height over the triangular projection.

Step 1: Geometry The projection is the triangle $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Above it, the solid runs from $z = 0$ to the sloping plane $z = 2 - x - y$.



Step 2: Evaluate $V = \int_0^1 \int_0^{1-x} \int_0^{2-x-y} 1 \, dz \, dy \, dx = \int_0^1 \left(\frac{3}{2} - 2x + \frac{x^2}{2} \right) dx = \boxed{\frac{2}{3}}.$

Verification On the projection, $2 - x - y \geq 1$, so every vertical bound is positive.

Question 3

Find the volume of the first-octant tetrahedron $x/2 + y/3 + z/6 \leq 1$.

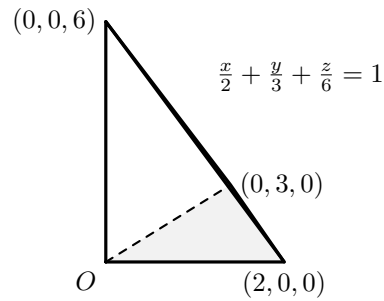
Tasks

1. Write bounds in the order $dz\,dy\,dx$.
2. Evaluate.
3. Verify geometrically.

Question 3 – Solution

Strategy. Solve the plane successively for z and for the projected y -boundary.

Step 1: Geometry The coordinate-plane faces meet the sloping face at the three axis intercepts shown.



Step 2: Bounds and integral $0 \leq x \leq 2$, $0 \leq y \leq 3 - 3x/2$, and $0 \leq z \leq 6 - 3x - 2y$. Hence $V = \int_0^2 \int_0^{3-3x/2} (6 - 3x - 2y) dy dx = \int_0^2 \frac{9}{4} (2 - x)^2 dx = \boxed{6}$.

Verification The intercept formula gives $V = (1/6)(2)(3)(6) = 6$.

Question 4

A box $0 \leq x \leq 2$, $0 \leq y \leq 1$, $0 \leq z \leq 3$ has density $\rho = 1 + x + z$.

Tasks

1. Find its mass.
2. Find the average density.
3. Check against the density range.

Question 4 – Solution

Strategy. Integrate density and divide by volume.

Step 1: Mass $M = \int_0^2 \int_0^1 \int_0^3 (1 + x + z) dz dy dx = \int_0^2 (15/2 + 3x) dx = \boxed{21}$.

Step 2: Average The volume is 6, so $\boxed{\rho_{\text{avg}} = 21/6 = 7/2}$.

Verification Here $1 \leq \rho \leq 6$; $7/2$ lies in this interval. Also the coordinate-average rule gives $1 + 1 + 3/2 = 7/2$.

Question 5

Find the average of $f = x^2 + y^2 + z^2$ on $[-1, 1]^3$.

Tasks

1. Use symmetry and separability.
2. Compute the average.
3. Compare with the range.

Question 5 – Solution

Strategy. The three squared-coordinate contributions are identical.

Step 1: Integral $\iiint_B x^2 dV = (\int_{-1}^1 x^2 dx)(2)(2) = 8/3$. Thus $\iiint_B f dV = 3(8/3) = 8$.

Step 2: Average Since $\text{vol}(B) = 8$, $\boxed{f_{\text{avg}} = 1}$.

Verification $0 \leq f \leq 3$, and each coordinate square has average $1/3$, so their sum has average 1 .

Question 6

Evaluate $\iiint_B (x^3 y z^2 + 4) \, dV$ on $B = [-2, 2] \times [-1, 1] \times [0, 3]$.

Tasks

1. Identify a cancellation symmetry.
2. Evaluate the integral.
3. State why the region matters.

Question 6 – Solution

Strategy. Pair x with $-x$; the nonconstant term is odd in x .

Step 1: Cancellation x^3yz^2 integrates to zero over the symmetric x -interval.

Step 2: Constant term The box volume is $(4)(2)(3) = 24$, so $\boxed{\iiint_B (x^3yz^2 + 4)dV = 4(24) = 96}$.

Verification Reflection $(x, y, z) \mapsto (-x, y, z)$ preserves B and reverses only the nonconstant term. Without that regional symmetry the cancellation would not follow.

Question 7

For $E = \{x, y, z \geq 0 : x + y + z \leq 1\}$, rewrite $\iiint_E z \, dV$ in the orders $dz \, dy \, dx$ and $dx \, dz \, dy$, then evaluate.

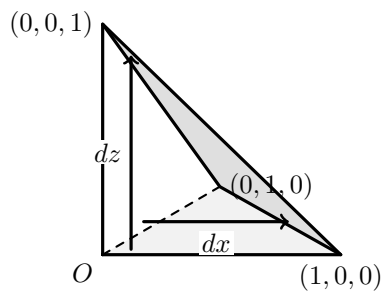
Tasks

1. Give both sets of bounds.
2. Evaluate one order.
3. Verify by symmetry.

Question 7 – Solution

Strategy. Each order peels coordinates from the simplex inequality.

Step 1: Geometry The simplex is bounded by the coordinate planes and $x + y + z = 1$. The two marked fibers show which coordinate is peeled off first.



Step 2: Orders $I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \int_0^1 \int_0^{1-y} \int_0^{1-y-z} z \, dx \, dz \, dy$. The first form gives $I = \frac{1}{6} \int_0^1 (1-x)^3 dx = \boxed{\frac{1}{24}}$.

Verification The simplex volume is $1/6$ and its centroid has $\bar{z} = 1/4$, so the moment is $(1/6)(1/4) = 1/24$.

Question 8

Let $I(a) = \int_0^1 \int_0^2 \int_0^3 (a + x - y + z) \, dz \, dy \, dx$. Find a so the average integrand is 5.

Tasks

1. Evaluate $I(a)$.
2. Solve the average condition.
3. Verify using coordinate averages.

Question 8 – Solution

Strategy. Convert average 5 into total 5 times box volume.

Step 1: Integral The volume is 6 and the coordinate averages are $(1/2, 1, 3/2)$, hence $I(a) = 6(a + 1)$.

Step 2: Solve $I(a)/6 = 5$ gives $a + 1 = 5$, so $\boxed{a = 4}$.

Verification At $a = 4$, the average is $4 + 1/2 - 1 + 3/2 = 5$, and the total integral is 30.

Question 9

For continuous f, g, h on their respective intervals, prove the product rule for $f(x)g(y)h(z)$ on $[a, b] \times [c, d] \times [e, k]$, then apply it to $(1+x)y^2e^z$ on $[0, 1] \times [-1, 1] \times [0, \ln 2]$.

Tasks

1. Prove factorization.
2. Evaluate the example.
3. Check the sign.

Question 9 – Solution

Strategy. Pull factors independent of the current integration variable outside each inner integral.

Step 1: Proof Successive integration gives $\iiint fgh \, dV = (\int_a^b f)(\int_c^d g)(\int_e^k h)$. Continuity guarantees the orders are valid.

Step 2: Example $\int_0^1 (1+x)dx = 3/2$, $\int_{-1}^1 y^2 dy = 2/3$, and $\int_0^{\ln 2} e^z dz = 1$. Their product is $\boxed{1}$.

Verification All factors are nonnegative on the box, matching the positive result.

Question 10

A solid E has volume 12, and $2 \leq f \leq 7$ throughout E . Another integrable function g satisfies $|g - f| \leq 0.05$. Given $\iiint_E f \, dV = 50$, bound $\iiint_E g \, dV$.

Tasks

1. Check consistency of the data.
2. Derive sharp bounds for the new integral.
3. Explain sharpness.

Question 10 – Solution

Strategy. Integrate pointwise inequalities and scale uniform error by volume.

Step 1: Consistency $24 \leq \iiint_E f \, dV \leq 84$, so 50 is possible.

Step 2: Perturbation From $-0.05 \leq g - f \leq 0.05$ and $\text{vol}(E) = 12$,

$$-0.6 \leq \iiint_E (g - f) \, dV \leq 0.6.$$

Therefore $\boxed{49.4 \leq \iiint_E g \, dV \leq 50.6}.$

Verification The endpoints occur for $g = f - 0.05$ and $g = f + 0.05$, so the supplied data guarantee no tighter interval.