

Question 1

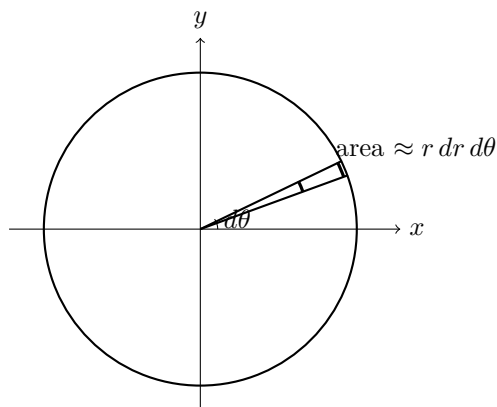
Let D be the disk $x^2 + y^2 \leq 4$.

Tasks

1. Convert $x^2 + y^2$ and dA to polar form.
2. Set up and evaluate $\iint_D (x^2 + y^2) dA$.
3. Explain the Jacobian factor geometrically.

Question 1 – Solution

Strategy. Radial symmetry turns the integrand into r^2 and the area element into $r \, dr \, d\theta$.



Step 1: Conversion The disk is $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$, and $x^2 + y^2 = r^2$. Hence

$$\iint_D (x^2 + y^2) dA = \int_0^{2\pi} \int_0^2 r^2 (r \, dr \, d\theta).$$

Step 2: Evaluate

$$\int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} 4 \, d\theta = \boxed{8\pi}.$$

Verification A thin polar cell has radial side dr and arc-length side approximately $r \, d\theta$, so its area is $r \, dr \, d\theta$. Omitting r would give the wrong units and value.

Question 2

Let D be the annular sector

$$1 \leq x^2 + y^2 \leq 9, \quad \frac{\pi}{6} \leq \arg(x + iy) \leq \frac{\pi}{2}.$$

Tasks

1. Write polar bounds.
2. Evaluate $\iint_D 1/\sqrt{x^2 + y^2} \, dA$.
3. Check finiteness and units.

Question 2 – Solution

Strategy. The factor $1/r$ cancels the polar Jacobian, leaving a rectangular integral in (r, θ) .

Step 1: Bounds The radial inequality gives $1 \leq r \leq 3$, while $\pi/6 \leq \theta \leq \pi/2$.

Step 2: Evaluate

$$\begin{aligned}\iint_D \frac{1}{\sqrt{x^2 + y^2}} dA &= \int_{\pi/6}^{\pi/2} \int_1^3 \frac{1}{r} (r \, dr \, d\theta) \\ &= \int_{\pi/6}^{\pi/2} 2 \, d\theta = \boxed{\frac{2\pi}{3}}.\end{aligned}$$

Verification The region stays at least one unit from the origin, so the integrand is bounded and no singularity occurs. After cancellation the radial length is 2 and the angular width is $\pi/3$.

Question 3

Find the area enclosed by the cardioid $r = 1 + \cos \theta$.

Tasks

1. Choose a complete nonduplicating angular interval.
2. Derive the polar area integral from a double integral.
3. Evaluate it exactly.

Question 3 – Solution

Strategy. Integrate 1 with radial bounds from the pole to the cardioid over one full turn.

Step 1: Bounds Since $1 + \cos \theta \geq 0$, one traversal is $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1 + \cos \theta$.

Step 2: Area integral

$$A = \int_0^{2\pi} \int_0^{1+\cos \theta} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} (1 + \cos \theta)^2 \, d\theta.$$

Step 3: Evaluate Over a full period, $\int \cos \theta = 0$ and $\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$. Thus

$$A = \frac{1}{2}(2\pi + 0 + \pi) = \boxed{\frac{3\pi}{2}}.$$

The radial bound vanishes only at $\theta = \pi$, so the interval neither omits nor duplicates positive-area portions.

Question 4

Find the area inside $r = 2 \cos \theta$ but outside the unit circle $r = 1$.

Tasks

1. Determine the angles where the radial bounds are ordered.
2. Set up and evaluate the area integral.
3. Interpret the boundary intersections.

Question 4 – Solution

Strategy. Solve $2 \cos \theta = 1$ to locate the two intersection rays, then integrate from the unit circle to the offset circle.

Step 1: Angles The curves meet when $\cos \theta = 1/2$, so $\theta = \pm\pi/3$. Between these angles, $2 \cos \theta \geq 1$.

Step 2: Area

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \int_1^{2 \cos \theta} r \, dr \, d\theta \\ &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4 \cos^2 \theta - 1) d\theta \\ &= \frac{1}{2} \left(4 \left(\frac{\pi}{3} + \frac{\sqrt{3}}{4} \right) - \frac{2\pi}{3} \right) = \boxed{\frac{\pi}{3} + \frac{\sqrt{3}}{2}}. \end{aligned}$$

Verification At both endpoint rays the inner and outer radii equal 1, so slice width vanishes. At $\theta = 0$ the slice runs from 1 to 2, confirming the order.

Question 5

For $a > 0$, evaluate

$$\iint_{x^2+y^2 \leq a^2} e^{-(x^2+y^2)} dA.$$

Tasks

1. Convert to polar coordinates.
2. Evaluate as a function of a .
3. Find the limit as $a \rightarrow \infty$ and verify monotonicity.

Question 5 – Solution

Strategy. The exponent and disk are radial; use $u = r^2$ after including the Jacobian.

Step 1: Set up

$$I(a) = \int_0^{2\pi} \int_0^a e^{-r^2} r \, dr \, d\theta.$$

Step 2: Evaluate With $u = r^2$, $du = 2r \, dr$,

$$I(a) = 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^a = \boxed{\pi(1 - e^{-a^2})}.$$

Step 3: Checks Since $I'(a) = 2\pi a e^{-a^2} > 0$, enlarging the disk increases the integral. Also

$$\boxed{\lim_{a \rightarrow \infty} I(a) = \pi}.$$

The result approaches a finite limit because the exponential decay dominates the growing circumference.

Question 6

Let D be the first-quadrant triangle $x \geq 0$, $y \geq 0$, $x + y \leq 2$.

Tasks

1. Convert the line boundary to a radial bound.
2. Use polar coordinates to compute the area.
3. Verify with elementary triangle geometry.

Question 6 – Solution

Strategy. A ray at angle θ meets $x + y = 2$ when $r(\cos \theta + \sin \theta) = 2$.

Step 1: Polar bounds First-quadrant rays give $0 \leq \theta \leq \pi/2$, and

$$0 \leq r \leq \frac{2}{\cos \theta + \sin \theta}.$$

Step 2: Area

$$A = \frac{1}{2} \int_0^{\pi/2} \frac{4}{(\cos \theta + \sin \theta)^2} d\theta.$$

Let $u = \tan \theta$. Since $(\cos \theta + \sin \theta)^2 = (1 + u)^2 / (1 + u^2)$ and $d\theta = du / (1 + u^2)$,

$$A = 2 \int_0^\infty \frac{du}{(1 + u)^2} = \boxed{2}.$$

Verification The Cartesian triangle has legs of length 2, so its area is $\frac{1}{2}(2)(2) = 2$. The nonconstant radial bound is essential; a constant bound would describe a sector, not the triangle.

Question 7

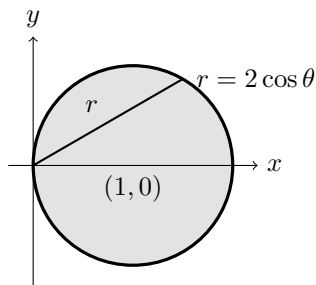
Let D be the disk $(x - 1)^2 + y^2 \leq 1$. Evaluate $\iint_D x \, dA$ in polar coordinates.

Tasks

1. Convert the offset circle to polar bounds.
2. Evaluate the integral.
3. Verify the result using area and symmetry.

Question 7 – Solution

Strategy. Rewrite the circle as $r = 2 \cos \theta$ and remember that $x = r \cos \theta$ contributes another factor of r with dA .



Step 1: Bounds The boundary equation becomes $r^2 - 2r \cos \theta = 0$. Thus

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 2 \cos \theta.$$

Step 2: Evaluate

$$\begin{aligned} \iint_D x \, dA &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 \cos \theta \, dr \, d\theta \\ &= \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta = \frac{8}{3} \cdot \frac{3\pi}{8} = \boxed{\pi}. \end{aligned}$$

Verification The disk has area π and is symmetric about the vertical line $x = 1$, so its average x -coordinate is 1. Hence its first moment is $1 \cdot \pi = \pi$.

Question 8

Find the area of one petal of the rose $r = \cos(2\theta)$ centered on the positive x -axis.

Tasks

1. Determine the angular interval for exactly that petal.
2. Set up and evaluate its area.
3. Explain why negative r values are excluded.

Question 8 – Solution

Strategy. The desired petal begins and ends where $\cos(2\theta) = 0$ around $\theta = 0$.

Step 1: Bounds The adjacent zeros are $\theta = \pm\pi/4$, and $\cos(2\theta) \geq 0$ between them. Thus $0 \leq r \leq \cos(2\theta)$.

Step 2: Area

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \cos^2(2\theta) d\theta \\ &= \frac{1}{4} \left[\theta + \frac{\sin 4\theta}{4} \right]_{-\pi/4}^{\pi/4} = \boxed{\frac{\pi}{8}}. \end{aligned}$$

Verification The sine endpoint terms vanish. Extending beyond these zeros introduces negative radial values, which trace different petals rather than enlarging the positive- x petal.

Question 9

An unknown annulus $a \leq r \leq b$, with $0 < a < b$, has area 3π and satisfies

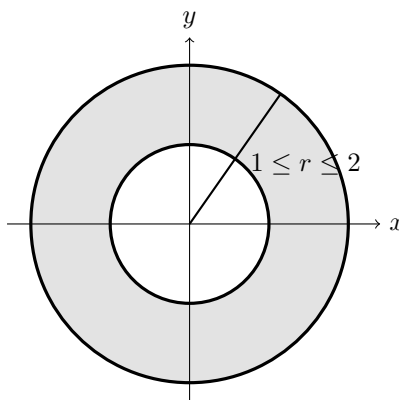
$$\iint_D (x^2 + y^2) dA = \frac{15\pi}{2}.$$

Tasks

1. Translate both conditions into equations for a^2, b^2 .
2. Recover the radii.
3. Verify both measurements.

Question 9 – Solution

Strategy. Set $A = a^2$ and $B = b^2$; area gives their difference, while the radial moment gives a difference of squares.



Step 1: Equations Area gives

$$\pi(b^2 - a^2) = 3\pi, \quad \text{so} \quad B - A = 3.$$

Also

$$\int_0^{2\pi} \int_a^b r^3 dr d\theta = \frac{\pi}{2}(b^4 - a^4) = \frac{15\pi}{2},$$

so $(B - A)(B + A) = 15$.

Step 2: Solve Since $B - A = 3$, we get $B + A = 5$. Therefore $B = 4$, $A = 1$, and positivity gives

$$\boxed{a = 1, \quad b = 2}.$$

Verification The area is $\pi(4 - 1) = 3\pi$, and the moment is $\frac{\pi}{2}(16 - 1) = 15\pi/2$. Both conditions hold exactly.

Question 10

Let D be the upper half-disk $x^2 + y^2 \leq 9$, $y \geq 0$. Evaluate

$$\iint_D (x^2 + y^2 + xy) dA.$$

Tasks

1. Convert the region and integrand to polar coordinates.
2. Use angular symmetry to eliminate the mixed term.
3. Evaluate the remaining integral and verify its sign.

Question 10 – Solution

Strategy. In polar form the radial term is r^2 , while $xy = r^2 \cos \theta \sin \theta$ cancels over $0 \leq \theta \leq \pi$.

Step 1: Set up

$$0 \leq \theta \leq \pi, \quad 0 \leq r \leq 3,$$

so

$$I = \int_0^\pi \int_0^3 r^3 (1 + \cos \theta \sin \theta) \, dr \, d\theta.$$

Step 2: Symmetry Since

$$\int_0^\pi \cos \theta \sin \theta \, d\theta = 0,$$

the xy contribution vanishes, matching reflection symmetry across the y -axis.

Step 3: Evaluate

$$I = \int_0^\pi \left[\frac{r^4}{4} \right]_0^3 d\theta = \int_0^\pi \frac{81}{4} d\theta = \boxed{\frac{81\pi}{4}}.$$

The surviving integrand $x^2 + y^2$ is nonnegative, so the positive result is necessary. Its scale is radius squared times half-disk area, as expected.