

Question 1

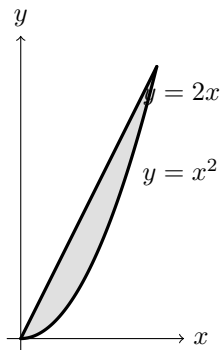
Let D be bounded by $y = x^2$ and $y = 2x$.

Tasks

1. Find the intersection points and describe D as a Type I region.
2. Set up and evaluate $\iint_D (x + y) dA$.
3. Verify that the bounds keep the lower curve below the upper curve.

Question 1 – Solution

Strategy. Solve the boundary intersection first, then use vertical slices from the parabola to the line.



Step 1: Region The curves meet where $x^2 = 2x$, so $x = 0, 2$. For $0 \leq x \leq 2$, $x^2 \leq 2x$. Thus

$$D = \{(x, y) : 0 \leq x \leq 2, x^2 \leq y \leq 2x\}.$$

Step 2: Integral

$$\begin{aligned} \iint_D (x + y) dA &= \int_0^2 \int_{x^2}^{2x} (x + y) dy dx \\ &= \int_0^2 \left[xy + \frac{y^2}{2} \right]_{x^2}^{2x} dx \\ &= \int_0^2 \left(4x^2 - x^3 - \frac{x^4}{2} \right) dx \\ &= \frac{32}{3} - 4 - \frac{16}{5} = \boxed{\frac{52}{15}}. \end{aligned}$$

Verification The slice height is $2x - x^2 = x(2 - x) \geq 0$ on $[0, 2]$. Also $x + y \geq 0$ throughout D , consistent with the positive answer.

Question 2

Let D be bounded by $x = y^2$ and $x = 2y$.

Tasks

1. Describe D as a Type II region.
2. Evaluate $\iint_D x \, dA$ using horizontal slices.
3. Check the sign and the intersection endpoints.

Question 2 – Solution

Strategy. Treat x as the inner variable and determine which curve is leftmost between the intersection values of y .

Step 1: Region From $y^2 = 2y$, the curves meet at $y = 0, 2$. On this interval, $y^2 \leq 2y$, so

$$D = \{(x, y) : 0 \leq y \leq 2, y^2 \leq x \leq 2y\}.$$

Step 2: Integral

$$\begin{aligned}\iint_D x \, dA &= \int_0^2 \int_{y^2}^{2y} x \, dx \, dy = \frac{1}{2} \int_0^2 (4y^2 - y^4) \, dy \\ &= \left[\frac{2y^3}{3} - \frac{y^5}{5} \right]_0^2 = \frac{16}{3} - \frac{16}{5} = \boxed{\frac{32}{15}}.\end{aligned}$$

Verification Both boundaries have $x \geq 0$ on $0 \leq y \leq 2$, so the integral of x must be nonnegative. The width $2y - y^2 = y(2 - y)$ vanishes at both intersection endpoints as the geometry requires.

Question 3

The region D lies between $y = x$ and $y = \sqrt{x}$ in the first quadrant.

Tasks

1. Describe D using vertical slices and using horizontal slices.
2. Write $\iint_D 1 \, dA$ in both orders.
3. Evaluate the area and verify the two descriptions agree.

Question 3 – Solution

Strategy. Solve each boundary for the inner variable and use the common endpoints $(0, 0)$ and $(1, 1)$.

Step 1: Two descriptions Since $x \leq \sqrt{x}$ on $[0, 1]$,

$$D = \{0 \leq x \leq 1, x \leq y \leq \sqrt{x}\}.$$

Solving for x , the parabola is $x = y^2$ and the line is $x = y$. For $0 \leq y \leq 1$, $y^2 \leq y$, so

$$D = \{0 \leq y \leq 1, y^2 \leq x \leq y\}.$$

Step 2: Both orders

$$\iint_D 1 \, dA = \int_0^1 \int_x^{\sqrt{x}} 1 \, dy \, dx = \int_0^1 \int_{y^2}^y 1 \, dx \, dy.$$

Step 3: Area The horizontal description gives

$$\text{area}(D) = \int_0^1 (y - y^2) \, dy = \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \boxed{\frac{1}{6}}.$$

Directly, $\int_0^1 (\sqrt{x} - x) \, dx = 2/3 - 1/2 = 1/6$, verifying the reversal.

Question 4

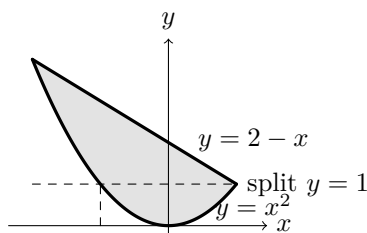
Let D be the region between $y = x^2$ and $y = 2 - x$.

Tasks

1. Describe D with vertical slices.
2. Reverse the order, explaining why horizontal slices require two integrals.
3. Compute the area in the reversed order and check it in the original order.

Question 4 – Solution

Strategy. Locate the intersection points and the height where the right boundary changes from the parabola to the line.



Step 1: Vertical description Solving $x^2 = 2 - x$ gives $x = -2, 1$. Hence

$$D = \{(x, y) : -2 \leq x \leq 1, x^2 \leq y \leq 2 - x\}.$$

Step 2: Reverse and split Horizontal slices begin at $x = -\sqrt{y}$. For $0 \leq y \leq 1$, they end at $x = \sqrt{y}$; for $1 \leq y \leq 4$, they end on the line $x = 2 - y$. Thus

$$\iint_D 1 \, dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1 \, dx \, dy + \int_1^4 \int_{-\sqrt{y}}^{2-y} 1 \, dx \, dy.$$

Step 3: Area

$$\begin{aligned} A &= \int_0^1 2\sqrt{y} \, dy + \int_1^4 (2 - y + \sqrt{y}) \, dy \\ &= \frac{4}{3} + \left[2y - \frac{y^2}{2} + \frac{2}{3}y^{3/2} \right]_1^4 = \frac{4}{3} + \frac{19}{6} = \boxed{\frac{9}{2}}. \end{aligned}$$

The original order gives $\int_{-2}^1 (2 - x - x^2) dx = 9/2$, confirming the split.

Question 5

Let

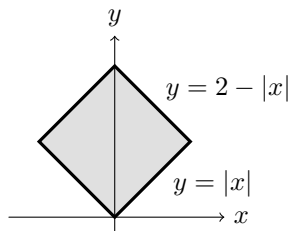
$$D = \{(x, y) : -1 \leq x \leq 1, |x| \leq y \leq 2 - |x|\}.$$

Tasks

1. Sketch or describe the region and identify its symmetry.
2. Evaluate $\iint_D y \, dA$ by splitting at $x = 0$ or using $|x|$ directly.
3. Find the area of D and hence the average value of y .

Question 5 – Solution

Strategy. Each vertical slice has endpoints symmetric about the horizontal line $y = 1$, making its average height immediate.



Step 1: Geometry The vertices are $(0,0), (1,1), (0,2), (-1,1)$. The diamond is symmetric about both $x = 0$ and $y = 1$.

Step 2: Integral of y

$$\begin{aligned}\iint_D y \, dA &= \int_{-1}^1 \int_{|x|}^{2-|x|} y \, dy \, dx \\ &= \frac{1}{2} \int_{-1}^1 ((2-|x|)^2 - |x|^2) \, dx \\ &= \int_{-1}^1 (2 - 2|x|) \, dx = \boxed{2}.\end{aligned}$$

Step 3: Area and average The slice width is $2 - 2|x|$, so

$$A = \int_{-1}^1 (2 - 2|x|) \, dx = 2.$$

Therefore $y_{\text{avg}} = 2/2 = \boxed{1}$, exactly the symmetry line. This independently checks the integral: total equals average times area.

Question 6

Consider

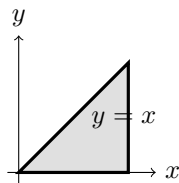
$$I = \int_0^1 \int_y^1 e^{x^2} dx dy.$$

Tasks

1. Describe the triangular region represented by the bounds.
2. Explain why the displayed order is not convenient for elementary integration.
3. Reverse the order and evaluate I exactly.

Question 6 – Solution

Strategy. Draw the triangle $0 \leq y \leq x \leq 1$; reversing the order creates an inner interval whose length supplies the factor needed for substitution.



Step 1: Region The displayed bounds say $0 \leq y \leq 1$ and $y \leq x \leq 1$. Equivalently,

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}.$$

Step 2: Why reverse In the given order, the inner antiderivative $\int e^{x^2} dx$ is not elementary. With dy first, e^{x^2} is constant in the inner variable.

Step 3: Reversed evaluation

$$\begin{aligned} I &= \int_0^1 \int_0^x e^{x^2} dy dx = \int_0^1 x e^{x^2} dx \\ &= \frac{1}{2} \int_0^1 e^u du = \left[\frac{e-1}{2} \right], \quad u = x^2. \end{aligned}$$

The integrand and region are positive, so the positive result is consistent. Differentiating $e^{x^2}/2$ returns $x e^{x^2}$.

Question 7

A solid lies above the triangle

$$D = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 1\}$$

and below the plane $z = 1 - x - y$.

Tasks

1. Write Cartesian bounds for D .
2. Evaluate the volume using a double integral.
3. Verify the answer using the geometry of a tetrahedron.

Question 7 – Solution

Strategy. Use vertical slices of the triangular base, then compare the integral with the intercept-form tetrahedron.

Step 1: Bounds For $0 \leq x \leq 1$, the variable y runs from 0 to $1 - x$. Therefore

$$V = \int_0^1 \int_0^{1-x} (1 - x - y) dy dx.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^1 \left[(1-x)y - \frac{y^2}{2} \right]_0^{1-x} dx \\ &= \frac{1}{2} \int_0^1 (1-x)^2 dx = \frac{1}{2} \left[x - x^2 + \frac{x^3}{3} \right]_0^1 \\ &= \boxed{\frac{1}{6}}. \end{aligned}$$

Step 3: Geometric check The coordinate planes and $x + y + z = 1$ enclose a tetrahedron with mutually perpendicular intercept lengths 1, 1, 1. Its volume is

$$\frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3} \left(\frac{1}{2} \right) (1) = \frac{1}{6}.$$

This matches the integral and verifies the top surface remains nonnegative over D .

Question 8

For $k > 0$, let D_k be the region enclosed by $y = x^2$ and $y = kx$.

Tasks

1. Find the intersections and write bounds for D_k .
2. Derive a formula for the area as a function of k .
3. Determine the unique $k > 0$ for which the area is $9/2$.

Question 8 – Solution

Strategy. Keep the line slope symbolic; the nonzero intersection becomes the outer endpoint of a vertical-slice area integral.

Step 1: Intersections and bounds From $x^2 = kx$,

$$x(x - k) = 0,$$

so the intersections occur at $x = 0, k$. Since $k > 0$, $x^2 \leq kx$ on $[0, k]$, and

$$D_k = \{(x, y) : 0 \leq x \leq k, x^2 \leq y \leq kx\}.$$

Step 2: Area formula

$$\begin{aligned} A(k) &= \int_0^k \int_{x^2}^{kx} 1 \, dy \, dx = \int_0^k (kx - x^2) \, dx \\ &= \left[\frac{kx^2}{2} - \frac{x^3}{3} \right]_0^k = \boxed{\frac{k^3}{6}}. \end{aligned}$$

Step 3: Inverse condition Set $k^3/6 = 9/2$. Then $k^3 = 27$, and the positive restriction gives

$$\boxed{k = 3}.$$

Substitution yields $A(3) = 27/6 = 9/2$. Since $A(k) = k^3/6$ is strictly increasing for $k > 0$, this positive solution is unique.

Question 9

Let

$$D = \{(x, y) : 0 \leq x \leq 2, 0 \leq y \leq \min(x, 2 - x)\}.$$

Tasks

1. Explain why the vertical-slice description must split at $x = 1$.
2. Give a single Type II description of D .
3. Evaluate $\iint_D (x + 2y) \, dA$ using the more efficient description.

Question 9 – Solution

Strategy. Recognize the triangle with vertices $(0, 0)$, $(1, 1)$, $(2, 0)$; horizontal slices use one left and one right boundary.

Step 1: Vertical split The smaller upper boundary is x on $[0, 1]$ and $2 - x$ on $[1, 2]$. Thus vertical integration would require

$$\int_0^1 \int_0^x (\cdots) dy dx + \int_1^2 \int_0^{2-x} (\cdots) dy dx.$$

Step 2: Single horizontal description For $0 \leq y \leq 1$, the line $y = x$ gives $x = y$ and $y = 2 - x$ gives $x = 2 - y$. Hence

$$D = \{(x, y) : 0 \leq y \leq 1, y \leq x \leq 2 - y\}.$$

Step 3: Evaluate

$$\begin{aligned} \iint_D (x + 2y) dA &= \int_0^1 \int_y^{2-y} (x + 2y) dx dy \\ &= \int_0^1 \left[\frac{x^2}{2} + 2yx \right]_y^{2-y} dy \\ &= \int_0^1 (2 + 2y - 4y^2) dy = \left[2y + y^2 - \frac{4}{3}y^3 \right]_0^1 \\ &= \boxed{\frac{5}{3}}. \end{aligned}$$

The integrand is nonnegative on D , agreeing with the sign.

Question 10

Let

$$D = \{(x, y) : 0 \leq y \leq 1, -y \leq x \leq y\}.$$

Tasks

1. Describe the region in the reverse order, including any required split.
2. Evaluate $\iint_D (x^2 + y) \, dA$ in the given order.
3. Use symmetry and the reversed description to check the structure of the answer.

Question 10 – Solution

Strategy. Use horizontal symmetry about the y -axis for evaluation, then express the lower boundary as $y = |x|$ when reversing.

Step 1: Reverse description The triangle has vertices $(0, 0)$, $(-1, 1)$, $(1, 1)$. In one compact form,

$$-1 \leq x \leq 1, \quad |x| \leq y \leq 1.$$

Without absolute values it splits into $-1 \leq x \leq 0$, $-x \leq y \leq 1$, and $0 \leq x \leq 1$, $x \leq y \leq 1$.

Step 2: Given order

$$\begin{aligned} \iint_D (x^2 + y) dA &= \int_0^1 \int_{-y}^y (x^2 + y) dx dy \\ &= \int_0^1 \left(\frac{2y^3}{3} + 2y^2 \right) dy \\ &= \frac{1}{6} + \frac{2}{3} = \boxed{\frac{5}{6}}. \end{aligned}$$

Step 3: Structural check In the reversed order, the two x -halves contribute equally because $x^2 + y$ is even in x . Thus

$$\iint_D (x^2 + y) dA = 2 \int_0^1 \int_x^1 (x^2 + y) dy dx.$$

Evaluating gives $2(5/12) = 5/6$. This verifies both the factor of two and the lower boundary $y = |x|$.