

Question 1

Evaluate the double integral of

$$f(x, y) = 3x^2y - 2x + 4$$

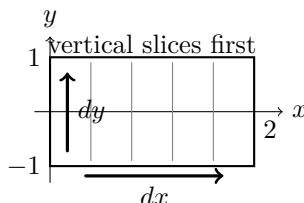
over $R = [0, 2] \times [-1, 1]$ by iterated integration.

Tasks

1. Evaluate by integrating with respect to y first.
2. Evaluate again by integrating with respect to x first.
3. Explain why the two answers must agree.

Question 1 – Solution

Strategy. Write both constant-bound orders explicitly and simplify the inner integral before performing the outer one.



Step 1: Integrate in y first

$$\begin{aligned}\int_0^2 \int_{-1}^1 (3x^2y - 2x + 4) dy dx &= \int_0^2 \left[\frac{3}{2}x^2y^2 - 2xy + 4y \right]_{-1}^1 dx \\ &= \int_0^2 (-4x + 8) dx \\ &= [-2x^2 + 8x]_0^2 = 8.\end{aligned}$$

The $3x^2y$ term vanishes because it is odd in y over a symmetric interval.

Step 2: Integrate in x first

$$\begin{aligned}\int_{-1}^1 \int_0^2 (3x^2y - 2x + 4) dx dy &= \int_{-1}^1 [x^3y - x^2 + 4x]_0^2 dy \\ &= \int_{-1}^1 (8y + 4) dy = 8.\end{aligned}$$

Thus

$$\boxed{\iint_R f dA = 8}.$$

Verification The polynomial is continuous on the closed rectangle, so Fubini's theorem permits either order. In the second order the odd term $8y$ cancels, leaving constant height 4 across a y -interval of length 2, again giving 8.

Question 2

Evaluate

$$\iint_R e^{2x} \cos y \, dA, \quad R = [0, \ln 3] \times \left[0, \frac{\pi}{3}\right].$$

Tasks

1. Recognize and use the product structure of the integrand.
2. Evaluate the integral in each order.
3. Verify the exact result by multiplying the two one-variable integrals.

Question 2 – Solution

Strategy. The integrand is a product of an x -only factor and a y -only factor, so constant rectangular bounds make the two integrations separate.

Step 1: Integrate in y first

$$\begin{aligned}\int_0^{\ln 3} \int_0^{\pi/3} e^{2x} \cos y \, dy \, dx &= \int_0^{\ln 3} e^{2x} [\sin y]_0^{\pi/3} \, dx \\ &= \frac{\sqrt{3}}{2} \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} \\ &= \frac{\sqrt{3}}{4} (9 - 1) = 2\sqrt{3}.\end{aligned}$$

Step 2: Reverse the order

$$\begin{aligned}\int_0^{\pi/3} \int_0^{\ln 3} e^{2x} \cos y \, dx \, dy &= \int_0^{\pi/3} \cos y \left[\frac{1}{2} e^{2x} \right]_0^{\ln 3} \, dy \\ &= 4[\sin y]_0^{\pi/3} = 2\sqrt{3}.\end{aligned}$$

Therefore

$$\boxed{\iint_R e^{2x} \cos y \, dA = 2\sqrt{3}}.$$

Verification Direct factorization gives

$$\left(\int_0^{\ln 3} e^{2x} \, dx \right) \left(\int_0^{\pi/3} \cos y \, dy \right) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3},$$

confirming both orders and the endpoint $e^{2\ln 3} = 9$.

Question 3

Consider

$$I = \int_0^1 \int_0^2 x e^{xy} dy dx.$$

Tasks

1. Evaluate I using the displayed order.
2. Write the reversed iterated integral and its inner antiderivative for $y \neq 0$.
3. Explain why the displayed order is substantially more efficient and check the behavior at $x = 0$.

Question 3 – Solution

Strategy. Treat x as constant in the inner integral; the factor x is exactly the derivative needed for e^{xy} with respect to y .

Step 1: Efficient order For $x \neq 0$,

$$\int_0^2 x e^{xy} dy = [e^{xy}]_0^2 = e^{2x} - 1.$$

At $x = 0$ the original integrand is identically zero and the same expression gives $e^0 - 1 = 0$, so the formula extends continuously. Hence

$$\begin{aligned} I &= \int_0^1 (e^{2x} - 1) dx = \left[\frac{1}{2} e^{2x} - x \right]_0^1 \\ &= \boxed{\frac{e^2 - 3}{2}}. \end{aligned}$$

Step 2: Reversed order The rectangle permits

$$I = \int_0^2 \int_0^1 x e^{xy} dx dy.$$

For $y \neq 0$, integration by parts gives

$$\int x e^{xy} dx = \frac{e^{xy}(xy - 1)}{y^2} + C,$$

so the inner result is

$$\frac{e^y(y - 1) + 1}{y^2}.$$

Step 3: Efficiency and verification The reversed expression has a removable singular appearance at $y = 0$ and requires another nonobvious antiderivative. The original order collapses immediately by substitution. Continuity of $x e^{xy}$ on the rectangle guarantees by Fubini's theorem that either completed order has the same value.

Question 4

A student is given

$$I = \int_{-1}^1 \int_0^2 (x - 2y) \, dx \, dy$$

and claims the inner integral is $[x^2/2 - 2xy]_{-1}^1$.

Tasks

1. Diagnose the student's error using the differential and bounds.
2. Evaluate the integral correctly as written.
3. Reverse the order on the same rectangle and verify the result.

Question 4 – Solution

Strategy. Read an iterated integral from the inside out: the inner differential determines both the variable of integration and which pair of bounds belongs to it.

Step 1: Diagnose The inner differential is dx , so x runs from 0 to 2 while y is held constant. The student incorrectly used the outer y -bounds, -1 and 1 , as x -bounds.

Step 2: Evaluate as written

$$\begin{aligned} I &= \int_{-1}^1 \left[\frac{x^2}{2} - 2yx \right]_0^2 dy = \int_{-1}^1 (2 - 4y) dy \\ &= [2y - 2y^2]_{-1}^1 = \boxed{4}. \end{aligned}$$

The odd contribution $-4y$ cancels over $[-1, 1]$.

Step 3: Reverse correctly Constant rectangular bounds reverse to

$$I = \int_0^2 \int_{-1}^1 (x - 2y) dy dx.$$

Then

$$I = \int_0^2 [xy - y^2]_{-1}^1 dx = \int_0^2 2x dx = [x^2]_0^2 = 4.$$

The agreement verifies both the order reversal and the association of each bound pair with its variable.

Question 5

Evaluate

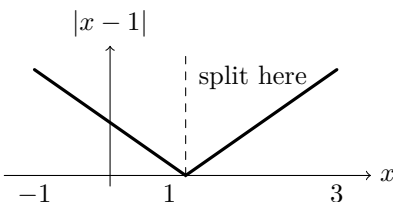
$$\iint_R |x - 1|(1 + y) \, dA, \quad R = [-1, 3] \times [0, 2].$$

Tasks

1. Choose an efficient order of integration.
2. Split at the correct location to remove the absolute value.
3. Evaluate the integral and its average value over R .

Question 5 – Solution

Strategy. Integrate the positive y -factor first, then split the remaining x -integral where $x - 1$ changes sign.



Step 1: Inner integral Using dy first,

$$\begin{aligned}\int_{-1}^3 \int_0^2 |x - 1|(1 + y) \, dy \, dx &= \int_{-1}^3 |x - 1| \left[y + \frac{y^2}{2} \right]_0^2 \, dx \\ &= 4 \int_{-1}^3 |x - 1| \, dx.\end{aligned}$$

Step 2: Absolute-value split

$$\begin{aligned}\int_{-1}^3 |x - 1| \, dx &= \int_{-1}^1 (1 - x) \, dx + \int_1^3 (x - 1) \, dx \\ &= 2 + 2 = 4.\end{aligned}$$

Therefore

$$\boxed{\iint_R |x - 1|(1 + y) \, dA = 16}.$$

Step 3: Average and check The rectangle has area $4 \cdot 2 = 8$, so

$$\boxed{f_{\text{avg}} = \frac{16}{8} = 2}.$$

The two x -pieces are mirror images about $x = 1$, and the nonnegative integrand confirms the positive sign.

Question 6

For a real parameter a , define

$$I(a) = \int_0^2 \int_{-1}^1 (axy^2 + x^2 - 2y) \, dy \, dx.$$

Tasks

1. Evaluate $I(a)$ exactly.
2. Find the unique value of a for which the integrand's average value on the rectangle is 1.
3. Verify the resulting integral directly from the inner accumulation function.

Question 6 – Solution

Strategy. Use parity in the symmetric y -interval, retain the parameter symbolically, and convert the prescribed average into a total integral.

Step 1: Inner accumulation Since $-2y$ is odd,

$$\begin{aligned}\int_{-1}^1 (axy^2 + x^2 - 2y) dy &= ax \left[\frac{y^3}{3} \right]_{-1}^1 + x^2[y]_{-1}^1 \\ &= \frac{2ax}{3} + 2x^2.\end{aligned}$$

Thus

$$I(a) = \int_0^2 \left(\frac{2ax}{3} + 2x^2 \right) dx = \frac{4a}{3} + \frac{16}{3}.$$

Step 2: Prescribed average The rectangle's area is $2 \cdot 2 = 4$. Average value 1 therefore requires $I(a) = 4$:

$$\frac{4a + 16}{3} = 4 \implies \boxed{a = -1}.$$

Step 3: Direct verification With $a = -1$, the inner accumulation is

$$-\frac{2x}{3} + 2x^2.$$

Consequently,

$$\int_0^2 \left(-\frac{2x}{3} + 2x^2 \right) dx = \left[-\frac{x^2}{3} + \frac{2x^3}{3} \right]_0^2 = -\frac{4}{3} + \frac{16}{3} = 4.$$

Dividing by area 4 returns the required average 1.

Question 7

A temperature field on a rectangular plate is

$$T(x, y) = 18 + 2x + y^2, \quad 0 \leq x \leq 3, \quad -1 \leq y \leq 1.$$

Tasks

1. Compute the average temperature using an iterated integral.
2. Compare it with the minimum and maximum temperatures on the plate.
3. Exhibit two distinct points where the temperature equals its average.

Question 7 – Solution

Strategy. Integrate the three terms separately, divide by plate area, and then solve the resulting level equation at convenient choices of y .

Step 1: Total and average The plate area is $A = 3 \cdot 2 = 6$. Its total temperature integral is

$$\begin{aligned}\int_0^3 \int_{-1}^1 (18 + 2x + y^2) dy dx &= 18(6) + \left(\int_0^3 2x dx \right) (2) + 3 \int_{-1}^1 y^2 dy \\ &= 108 + 18 + 2 = 128.\end{aligned}$$

Hence

$$T_{\text{avg}} = \frac{128}{6} = \frac{64}{3}.$$

Step 2: Range check The minimum is $T(0, 0) = 18$. The maximum 25 occurs at $(3, 1)$ and $(3, -1)$. Since

$$18 < \frac{64}{3} < 25,$$

the average lies strictly inside the temperature range, as required.

Step 3: Points at the average Solve

$$18 + 2x + y^2 = \frac{64}{3}.$$

Taking $y = 0$ gives $x = 5/3$, while taking $y = 1$ gives $x = 7/6$. Thus two examples are

$$\left(\frac{5}{3}, 0 \right) \quad \text{and} \quad \left(\frac{7}{6}, 1 \right).$$

Both lie on the plate, and substitution gives $64/3$ at each.

Question 8

Let p be continuous on $[a, b]$ and q continuous on $[c, d]$.

Tasks

1. Prove using iterated integrals that

$$\iint_{[a,b] \times [c,d]} p(x)q(y) \, dA = \left(\int_a^b p(x) \, dx \right) \left(\int_c^d q(y) \, dy \right).$$

2. Explain why the same proof works in either order.
3. Apply the result to $(1 + x^2) \cos y$ on $[-1, 1] \times [-\pi/2, \pi/2]$.

Question 8 – Solution

Strategy. In the inner integral, the factor depending only on the outer variable is constant and can be pulled outside.

Step 1: Factorization proof Integrating in y first gives

$$\begin{aligned}\int_a^b \int_c^d p(x)q(y) dy dx &= \int_a^b p(x) \left(\int_c^d q(y) dy \right) dx \\ &= \left(\int_a^b p(x) dx \right) \left(\int_c^d q(y) dy \right).\end{aligned}$$

The bracketed q -integral is a number, so it is constant in the outer x -integration.

Step 2: Other order Reversing the order first produces the constant $\int_a^b p(x) dx$ and then the q -integral. Continuity on the closed rectangle guarantees Fubini's theorem applies, so both forms represent the same double integral.

Step 3: Application Here

$$\int_{-1}^1 (1+x^2) dx = \left[x + \frac{x^3}{3} \right]_{-1}^1 = \frac{8}{3}$$

and

$$\int_{-\pi/2}^{\pi/2} \cos y dy = [\sin y]_{-\pi/2}^{\pi/2} = 2.$$

Therefore

$$\boxed{\iint_R (1+x^2) \cos y dA = \frac{16}{3}}.$$

Both factors are nonnegative on their intervals, verifying the sign.

Question 9

Evaluate

$$\iint_R (x^3 \cos y + x^2 \sin y + x^2) dA, \quad R = [-2, 2] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Tasks

1. Integrate with respect to y first and identify the term eliminated by y -symmetry.
2. Use x -symmetry in the outer integral.
3. Evaluate the surviving contribution and verify its sign.

Question 9 – Solution

Strategy. Use parity at each stage: the symmetric inner interval cancels the sine term, and the symmetric outer interval cancels the remaining odd power of x .

Step 1: Inner y -integration

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} (x^3 \cos y + x^2 \sin y + x^2) dy &= x^3 [\sin y]_{-\pi/2}^{\pi/2} - x^2 [\cos y]_{-\pi/2}^{\pi/2} + \pi x^2 \\ &= 2x^3 + \pi x^2.\end{aligned}$$

The $x^2 \sin y$ term vanishes because sine is odd in y .

Step 2: Outer x -integration Now $2x^3$ is odd on $[-2, 2]$, so it contributes zero. Therefore

$$\begin{aligned}\iint_R f dA &= \pi \int_{-2}^2 x^2 dx = \pi \left[\frac{x^3}{3} \right]_{-2}^2 \\ &= \boxed{\frac{16\pi}{3}}.\end{aligned}$$

Step 3: Verification The only surviving term is $x^2 \geq 0$, integrated over a y -interval of length π . Thus a positive result is necessary. Its one-variable integral is $16/3$, so multiplication by π confirms the scale and constant.

Question 10

Over the square $R = [-1, 1] \times [-1, 1]$, a solid lies between

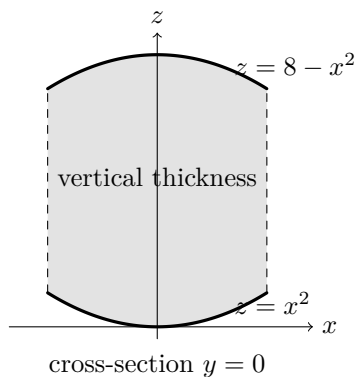
$$z_{\text{top}} = 8 - x^2 - y^2 \quad \text{and} \quad z_{\text{bottom}} = x^2 + y^2.$$

Tasks

1. Verify that the named top surface stays above the bottom surface on all of R .
2. Set up and evaluate an iterated integral for the volume.
3. Check the answer using symmetry and separate one-variable integrals.

Question 10 – Solution

Strategy. Integrate vertical thickness over the rectangular base, after confirming that the thickness never changes sign.



Step 1: Thickness The vertical thickness is

$$h(x, y) = 8 - 2x^2 - 2y^2.$$

On R , $x^2 + y^2 \leq 2$, so $h \geq 8 - 4 = 4 > 0$. Thus no top-bottom reversal or absolute value is needed.

Step 2: Iterated integral

$$\begin{aligned} V &= \int_{-1}^1 \int_{-1}^1 (8 - 2x^2 - 2y^2) dy dx \\ &= \int_{-1}^1 \left(16 - 4x^2 - \frac{4}{3} \right) dx \\ &= \left[\frac{44}{3}x - \frac{4}{3}x^3 \right]_{-1}^1 = \boxed{\frac{80}{3}}. \end{aligned}$$

Step 3: Independent check The constant term contributes $8 \text{ area}(R) = 32$. Also,

$$\iint_R x^2 dA = \left(\int_{-1}^1 x^2 dx \right) (2) = \frac{4}{3},$$

and identically $\iint_R y^2 dA = 4/3$. Hence

$$V = 32 - 2 \left(\frac{4}{3} \right) - 2 \left(\frac{4}{3} \right) = 32 - \frac{16}{3} = \frac{80}{3},$$

confirming the iterated calculation.