

Question 1

Let $a > 0$. Find the area enclosed by the astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

using the change of variables $x = u^3$, $y = v^3$.

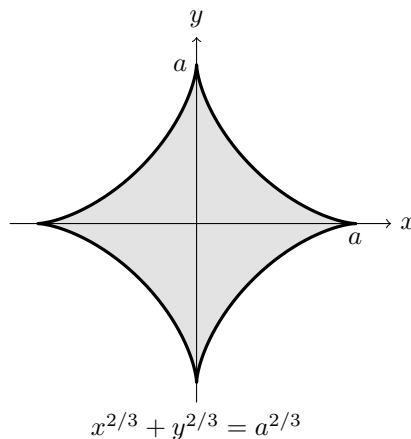
Tasks

1. Describe the transformed region and justify the substitution.
2. Compute the Jacobian and evaluate the area.
3. Check the scaling of the answer with respect to a .

Question 1 – Solution

Strategy. The cubic substitution converts the astroid into a disk, after which polar coordinates handle the transformed integral.

Step 1: Transform the region



Because the real cube function is one-to-one,

$$x^{2/3} = u^2, \quad y^{2/3} = v^2.$$

Thus the interior maps one-to-one to $u^2 + v^2 \leq a^{2/3}$, a disk of radius $a^{1/3}$.

Step 2: Jacobian and integral The absolute Jacobian is

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = |(3u^2)(3v^2)| = 9u^2v^2.$$

Writing $u = r \cos \theta$, $v = r \sin \theta$ gives

$$\begin{aligned} A &= 9 \int_0^{2\pi} \int_0^{a^{1/3}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \\ &= 9 \left(\frac{a^2}{6} \right) \left(\frac{\pi}{4} \right) = \boxed{\frac{3\pi a^2}{8}}. \end{aligned}$$

Verification Replacing a by ka enlarges every length by k , so area must scale by k^2 . The result is proportional to a^2 , as required.

Question 2

Find the area common to the disks

$$x^2 + y^2 \leq 2x, \quad x^2 + y^2 \leq 2y.$$

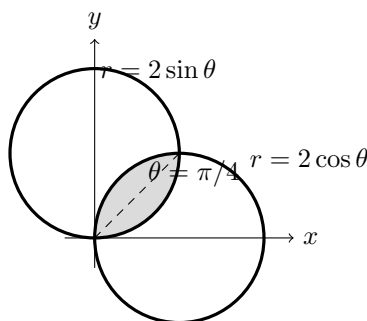
Tasks

1. Translate both inequalities into polar coordinates.
2. Identify where the radial boundary changes and evaluate the area.
3. Verify the symmetry used in the calculation.

Question 2 – Solution

Strategy. In the first quadrant the smaller of the two polar bounds switches at the line $\theta = \pi/4$.

Step 1: Polar description



For $r \geq 0$, the disks become

$$r \leq 2 \cos \theta, \quad r \leq 2 \sin \theta.$$

Their common region lies in $0 \leq \theta \leq \pi/2$. The bounds are equal at $\theta = \pi/4$.

Step 2: Evaluate Reflection across $y = x$ makes the two halves congruent, so

$$\begin{aligned} A &= 2 \int_0^{\pi/4} \int_0^{2 \sin \theta} r \, dr \, d\theta = 4 \int_0^{\pi/4} \sin^2 \theta \, d\theta \\ &= 4 \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{\pi/4} = \boxed{\frac{\pi}{2} - 1}. \end{aligned}$$

Verification The circles have equal radius and centers $(1, 0)$ and $(0, 1)$, which are interchanged by reflecting across $y = x$. Their intersections, $(0, 0)$ and $(1, 1)$, also lie on that line.

Question 3

Find the volume enclosed between the paraboloids

$$z = x^2 + y^2 \quad \text{and} \quad z = 6 - x^2 - y^2.$$

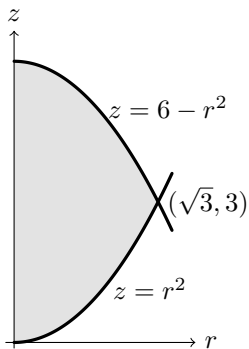
Tasks

1. Find the curve where the surfaces meet.
2. Set up and evaluate a cylindrical-coordinate integral.
3. Check that the height is nonnegative throughout the base.

Question 3 – Solution

Strategy. Subtract the lower paraboloid from the upper one over their circular projection.

Step 1: Intersection



Setting the surfaces equal gives

$$r^2 = 6 - r^2, \quad r = \sqrt{3}, \quad z = 3.$$

Hence the projection is the disk $0 \leq r \leq \sqrt{3}$.

Step 2: Volume The vertical height is $(6 - r^2) - r^2 = 6 - 2r^2$. Therefore

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\sqrt{3}} (6 - 2r^2)r \, dr \, d\theta \\ &= 2\pi \left[3r^2 - \frac{r^4}{2} \right]_0^{\sqrt{3}} = 2\pi \left(9 - \frac{9}{2} \right) = \boxed{9\pi}. \end{aligned}$$

Verification On $0 \leq r \leq \sqrt{3}$, the height is $2(3 - r^2) \geq 0$; it decreases from 6 at the axis to 0 at the intersection circle, matching the geometry.

Question 4

Let $a > 0$. Two perpendicular circular cylinders form the solid

$$x^2 + z^2 \leq a^2, \quad y^2 + z^2 \leq a^2.$$

Find its volume.

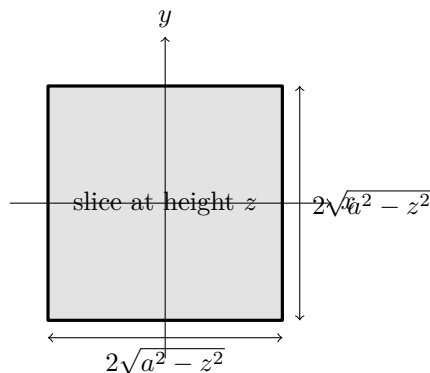
Tasks

1. Describe a horizontal cross-section at a fixed height z .
2. Integrate the cross-sectional area.
3. Check symmetry and dimensional scaling.

Question 4 – Solution

Strategy. At each fixed z , the two cylinder inequalities give independent bounds on x and y , producing a square.

Step 1: Cross-section



For $-a \leq z \leq a$,

$$|x| \leq \sqrt{a^2 - z^2}, \quad |y| \leq \sqrt{a^2 - z^2}.$$

Thus the slice is a square with area

$$A(z) = \left(2\sqrt{a^2 - z^2}\right)^2 = 4(a^2 - z^2).$$

Step 2: Integrate

$$\begin{aligned} V &= \int_{-a}^a 4(a^2 - z^2) dz = 8 \int_0^a (a^2 - z^2) dz \\ &= 8 \left[a^2 z - \frac{z^3}{3} \right]_0^a = \boxed{\frac{16a^3}{3}}. \end{aligned}$$

Verification The cross-sectional area is even in z , justifying the factor 2. The answer is proportional to a^3 , the correct scaling for a volume.

Question 5

A cylindrical hole of radius a is drilled through the center of a sphere of radius R , where $0 < a < R$. The remaining solid has height

$$h = 2\sqrt{R^2 - a^2}.$$

Find its volume and express the result using only h .

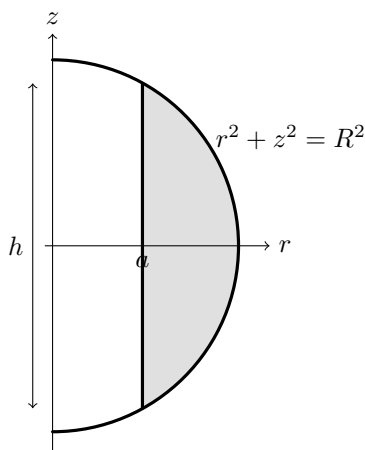
Tasks

1. Set up the volume in cylindrical coordinates.
2. Evaluate it and eliminate R and a in favor of h .
3. Explain the resulting “napkin-ring” invariance.

Question 5 – Solution

Strategy. Use cylindrical shells with $a \leq r \leq R$; the sphere supplies the upper and lower z -bounds.

Step 1: Bounds



The sphere is $r^2 + z^2 \leq R^2$. Hence

$$0 \leq \theta \leq 2\pi, \quad a \leq r \leq R, \quad -\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2}.$$

Step 2: Evaluate

$$\begin{aligned} V &= \int_0^{2\pi} \int_a^R 2\sqrt{R^2 - r^2} r \, dr \, d\theta \\ &= 4\pi \left[-\frac{(R^2 - r^2)^{3/2}}{3} \right]_a^R = \frac{4\pi}{3} (R^2 - a^2)^{3/2}. \end{aligned}$$

Since $R^2 - a^2 = h^2/4$,

$$\boxed{V = \frac{\pi h^3}{6}}.$$

Verification The final formula contains only the remaining height. Thus every drilled sphere with the same height h leaves the same volume, regardless of its original R and hole radius a .

Question 6

Find the volume of the solid torus

$$\left(\sqrt{x^2 + y^2} - 3\right)^2 + z^2 \leq 1$$

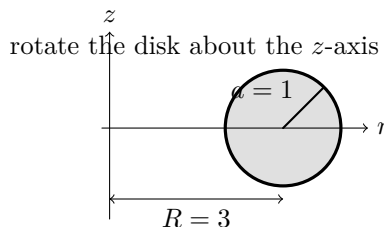
using cylindrical coordinates.

Tasks

1. Describe the bounds in the meridian (r, z) -plane.
2. Evaluate the triple integral, using symmetry where helpful.
3. Give a geometric check of the answer.

Question 6 – Solution

Strategy. In cylindrical coordinates the torus is obtained by rotating a radius-1 disk centered at $(r, z) = (3, 0)$.

Step 1: Bounds

The radial variable ranges from 2 to 4, and

$$-\sqrt{1 - (r - 3)^2} \leq z \leq \sqrt{1 - (r - 3)^2}, \quad 0 \leq \theta \leq 2\pi.$$

Step 2: Evaluate With $u = r - 3$,

$$\begin{aligned} V &= 2\pi \int_2^4 2r \sqrt{1 - (r - 3)^2} dr \\ &= 4\pi \int_{-1}^1 (u + 3) \sqrt{1 - u^2} du. \end{aligned}$$

The u -term is odd, while $\int_{-1}^1 \sqrt{1 - u^2} du = \pi/2$. Therefore

$$V = 4\pi(3) \frac{\pi}{2} = \boxed{6\pi^2}.$$

Verification The generating disk has area π , and its center travels a circle of circumference $2\pi(3) = 6\pi$. Their product is $6\pi^2$, agreeing with the integral.

Question 7

Consider the solids

$$S_1 = \{(x, y, z) : 0 \leq z \leq 1, x^2 + y^2 \leq z^2\}$$

and

$$S_2 = \{(x, y, z) : 0 \leq z \leq 1, 0 \leq x \leq z^2, 0 \leq y \leq \pi\}.$$

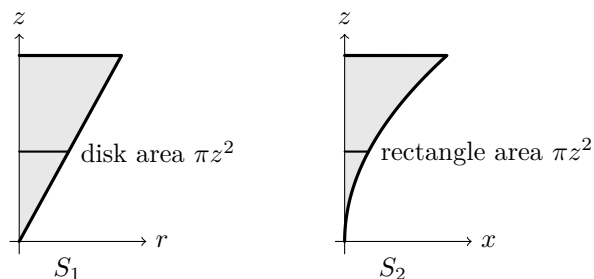
Tasks

1. Find the area of each horizontal cross-section.
2. Use Cavalieri's principle to compare the volumes.
3. Compute their common volume.

Question 7 – Solution

Strategy. Although the slices have different shapes, compare their areas at the same height.

Step 1: Slice areas



At height z , the slice of S_1 is a disk of radius z , so

$$A_1(z) = \pi z^2.$$

The slice of S_2 is a rectangle of side lengths z^2 and π , so

$$A_2(z) = z^2 \pi = \pi z^2.$$

Step 2: Compare and integrate Both solids occupy $0 \leq z \leq 1$ and have equal cross-sectional area at every height. Cavalieri's principle therefore gives equal volumes, and

$$V_1 = V_2 = \int_0^1 \pi z^2 dz = \pi \left[\frac{z^3}{3} \right]_0^1 = \boxed{\frac{\pi}{3}}.$$

Verification The conclusion depends on slice areas, not slice shapes: circular slices in S_1 and rectangular slices in S_2 accumulate the same volume height by height.

Question 8

For $A > 0$, let E_A be the solid bounded below by

$$z = x^2 + y^2$$

and above by the plane $z = A$. If E_A has volume 18π , determine A , the base radius, and the height.

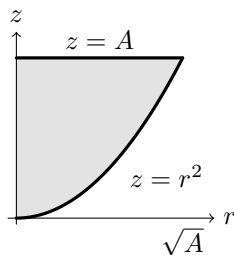
Tasks

1. Derive a formula for $V(A)$.
2. Solve the inverse volume condition.
3. Verify the dimensions and positivity of the result.

Question 8 – Solution

Strategy. First express the volume as a function of the unknown cutting height A , then solve the resulting equation.

Step 1: Geometry and volume



The plane and paraboloid meet when $r^2 = A$, so $0 \leq r \leq \sqrt{A}$. Thus

$$\begin{aligned} V(A) &= \int_0^{2\pi} \int_0^{\sqrt{A}} (A - r^2)r \, dr \, d\theta \\ &= 2\pi \left[\frac{Ar^2}{2} - \frac{r^4}{4} \right]_0^{\sqrt{A}} = \frac{\pi A^2}{2}. \end{aligned}$$

Step 2: Recover the dimensions The condition $V(A) = 18\pi$ yields

$$\frac{\pi A^2}{2} = 18\pi, \quad A^2 = 36.$$

Since $A > 0$,

$$\boxed{A = 6}, \quad \boxed{\text{base radius} = \sqrt{6}}, \quad \boxed{\text{height} = 6}.$$

Verification Substitution gives $V(6) = \pi(36)/2 = 18\pi$. The positive root is required because A is a height, and $r = \sqrt{A}$ follows from the intersection equation.

Question 9

Find the volume of the ellipsoidal shell

$$1 \leq \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} \leq 4.$$

Use the change of variables

$$x = 2u, \quad y = 3v, \quad z = 4w.$$

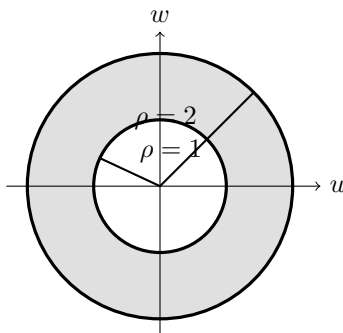
Tasks

1. Describe the transformed shell.
2. Compute the three-dimensional Jacobian and the volume.
3. Check the scaling factor geometrically.

Question 9 – Solution

Strategy. The coordinate scaling turns the ellipsoids into concentric spheres and multiplies every volume by a constant Jacobian.

Step 1: Transform the shell



scaled spherical shell

The inequality becomes

$$1 \leq u^2 + v^2 + w^2 \leq 4,$$

so the transformed region is the spherical shell $1 \leq \rho \leq 2$.

Step 2: Jacobian and volume The derivative matrix is diagonal, so

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = |(2)(3)(4)| = 24.$$

Consequently,

$$V = 24 \left[\frac{4\pi}{3} (2^3 - 1^3) \right] = 24 \left(\frac{28\pi}{3} \right) = \boxed{224\pi}.$$

Verification The map stretches lengths by factors 2, 3, and 4 along the coordinate axes, so it must stretch volume by their product 24, exactly the Jacobian used above.

Question 10

A sphere of radius R has surface-area-to-volume ratio

$$\frac{S}{V} = \frac{3}{2}.$$

Determine its radius, surface area, and volume.

Tasks

1. Derive V with a spherical-coordinate triple integral.
2. Derive S by viewing the two hemispheres as graphs.
3. Use the given ratio and verify its units.

Question 10 – Solution

Strategy. Rebuild both standard sphere formulas from integration, then use their ratio to recover the radius.

Step 1: Volume In spherical coordinates,

$$V = \int_0^{2\pi} \int_0^\pi \int_0^R \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = 2\pi(2) \frac{R^3}{3} = \frac{4\pi R^3}{3}.$$

Step 2: Surface area For the upper graph $z = \sqrt{R^2 - r^2}$, the area factor is $R/\sqrt{R^2 - r^2}$. Its area is

$$S_{\text{upper}} = \int_0^{2\pi} \int_0^R \frac{Rr}{\sqrt{R^2 - r^2}} \, dr \, d\theta = 2\pi R^2.$$

Doubling for both hemispheres gives $S = 4\pi R^2$.

Step 3: Use the ratio

$$\frac{S}{V} = \frac{4\pi R^2}{4\pi R^3/3} = \frac{3}{R} = \frac{3}{2}, \quad \boxed{R = 2}.$$

Therefore

$$\boxed{S = 16\pi}, \quad \boxed{V = \frac{32\pi}{3}}.$$

Verification Surface area divided by volume has units of inverse length, and $3/R$ has those units. Substituting $R = 2$ reproduces the stated ratio.