

Question 1

Let

$$R = [0, 2] \times [0, 1], \quad f(x, y) = 6 - x^2 - y.$$

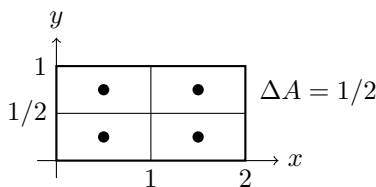
Partition R into two equal columns and two equal rows, and use the midpoint of each subrectangle as its sample point.

Tasks

1. List the four sample points and the common subrectangle area.
2. Compute the midpoint Riemann sum.
3. Interpret the sum geometrically and check it against bounds from the minimum and maximum of f on R .

Question 1 – Solution

Strategy. Translate the partition into four base areas, evaluate the height at each midpoint, and add the resulting prism volumes.



Step 1: Partition data Here $\Delta x = 1$, $\Delta y = 1/2$, and

$$\Delta A = \Delta x \Delta y = \frac{1}{2}.$$

The midpoints are

$$\left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{3}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{2}, \frac{3}{4}\right).$$

Step 2: Heights and sum In the same order, the four function values are

$$\frac{11}{2}, \quad \frac{7}{2}, \quad 5, \quad 3.$$

Therefore

$$S = \frac{1}{2} \left(\frac{11}{2} + \frac{7}{2} + 5 + 3 \right) = \boxed{\frac{17}{2}}.$$

Step 3: Interpretation and bounds The sum is the total volume of four midpoint prisms. On R , f decreases as either x or y increases, so

$$1 = f(2, 1) \leq f(x, y) \leq f(0, 0) = 6.$$

Since $\text{area}(R) = 2$, every Riemann-sum estimate lies between 2 and 12; indeed $17/2$ does. This also verifies that every sampled height was positive and on the correct scale.

Question 2

On $R = [0, 1] \times [0, 2]$, let $f(x, y) = x + 2y$. Divide the x -interval into m equal parts and the y -interval into n equal parts. Use the upper-right corner of each subrectangle as its sample point.

Tasks

1. Write the resulting double Riemann sum $S_{m,n}$.
2. Simplify the finite sum exactly using summation formulas.
3. Take the limit as $m, n \rightarrow \infty$ to evaluate the double integral from its definition.

Question 2 – Solution

Strategy. Express every sample point and area in terms of its indices, then separate the finite sums algebraically before taking the two-parameter limit.

Step 1: Grid and samples We have

$$\Delta x = \frac{1}{m}, \quad \Delta y = \frac{2}{n}, \quad \Delta A = \frac{2}{mn},$$

and the upper-right sample in cell (i, j) is $(i/m, 2j/n)$. Thus

$$S_{m,n} = \sum_{i=1}^m \sum_{j=1}^n \left(\frac{i}{m} + \frac{4j}{n} \right) \frac{2}{mn}.$$

Step 2: Exact finite sum Using $\sum_{k=1}^N k = N(N+1)/2$,

$$\begin{aligned} S_{m,n} &= \frac{2}{m^2 n} \sum_{i=1}^m \sum_{j=1}^n i + \frac{8}{mn^2} \sum_{i=1}^m \sum_{j=1}^n j \\ &= \frac{2}{m^2 n} \left(n \frac{m(m+1)}{2} \right) + \frac{8}{mn^2} \left(m \frac{n(n+1)}{2} \right) \\ &= \frac{m+1}{m} + 4 \frac{n+1}{n}. \end{aligned}$$

Step 3: Definition limit Both correction terms vanish independently, so

$$\boxed{\iint_R (x + 2y) dA = \lim_{m,n \rightarrow \infty} S_{m,n} = 1 + 4 = 5}.$$

The answer lies between $0 \cdot 2$ and $5 \cdot 2 = 10$, the bounds obtained from the extreme values of f on the area-2 rectangle.

Question 3

Let $f(x, y) = xy$ on the unit square $R = [0, 1] \times [0, 1]$. Partition R into an $n \times n$ uniform grid.

Tasks

1. Compute the lower sum L_n and upper sum U_n .
2. Use them to determine $\iint_R xy \, dA$ without iterated integration.
3. Find the least integer n for which the gap $U_n - L_n$ is less than 0.01.

Question 3 – Solution

Strategy. Because xy increases in each coordinate on the unit square, lower-left and upper-right corners produce the lower and upper sums.

Step 1: Lower sum Every cell has area $1/n^2$. Using indices $i, j = 0, \dots, n-1$ at lower-left corners,

$$\begin{aligned} L_n &= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{i}{n} \frac{j}{n} = \frac{1}{n^4} \left(\sum_{i=0}^{n-1} i \right)^2 \\ &= \frac{1}{n^4} \left(\frac{n(n-1)}{2} \right)^2 = \frac{(n-1)^2}{4n^2}. \end{aligned}$$

Step 2: Upper sum and squeeze Upper-right corners give

$$U_n = \frac{1}{n^4} \left(\sum_{i=1}^n i \right)^2 = \frac{(n+1)^2}{4n^2}.$$

Both sums approach $1/4$. Since $L_n \leq \iint_R xy \, dA \leq U_n$,

$$\boxed{\iint_R xy \, dA = \frac{1}{4}}.$$

Step 3: Resolution The uncertainty is

$$U_n - L_n = \frac{(n+1)^2 - (n-1)^2}{4n^2} = \frac{1}{n}.$$

Thus $1/n < 0.01$ requires $n > 100$, and the least integer is n=101. The strict inequality rules out $n = 100$.

Question 4

On $R = [-2, 2] \times [0, 3]$, define

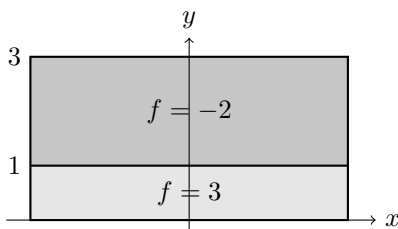
$$f(x, y) = \begin{cases} 3, & 0 \leq y \leq 1, \\ -2, & 1 < y \leq 3. \end{cases}$$

Tasks

1. Evaluate $\iint_R f \, dA$ by signed volume.
2. Evaluate $\iint_R |f| \, dA$ and explain the difference.
3. Find the average value of f on R .

Question 4 – Solution

Strategy. Split the base into the two constant-height bands and distinguish algebraic volume from total unsigned volume.



Step 1: Signed volume The lower band has area $4(1) = 4$ and height 3, so it contributes 12. The upper band has area $4(2) = 8$ and height -2 , so it contributes -16 . Hence

$$\iint_R f \, dA = 12 - 16 = -4.$$

Step 2: Unsigned volume Absolute value changes the second height from -2 to 2. Therefore

$$\iint_R |f| \, dA = 12 + 16 = 28.$$

The first integral records volume below the xy -plane negatively; the second records all geometric volume positively.

Step 3: Average value Since $\text{area}(R) = 4 \cdot 3 = 12$,

$$f_{\text{avg}} = \frac{1}{12} \iint_R f \, dA = -\frac{1}{3}.$$

As a check, $-1/3$ lies between the only two function values, -2 and 3 , and the wider negative band explains its negative sign.

Question 5

Let $R = [-1, 2] \times [0, 2]$. A continuous function f satisfies

$$2 \leq f(x, y) \leq 5, \quad \iint_R f \, dA = 19.$$

A second integrable function g satisfies $|g(x, y) - f(x, y)| \leq 0.1$ throughout R .

Tasks

1. Check that the stated integral of f is consistent with its pointwise bounds.
2. Evaluate $\iint_R (3f - 4) \, dA$.
3. Give the sharp bounds guaranteed by the data for $\iint_R g \, dA$.

Question 5 – Solution

Strategy. Convert pointwise inequalities to integral inequalities using the area of the rectangle, and use linearity for the transformed function.

Step 1: Comparison bounds The rectangle has area

$$A = (2 - (-1))(2 - 0) = 6.$$

Integrating $2 \leq f \leq 5$ over R gives

$$12 \leq \iint_R f \, dA \leq 30.$$

The given value 19 lies in this interval, so the data are consistent.

Step 2: Linearity Because the integral of the constant 4 is $4A = 24$,

$$\iint_R (3f - 4) \, dA = 3 \iint_R f \, dA - \iint_R 4 \, dA = 57 - 24 = \boxed{33}.$$

Step 3: Stability bound From $|g - f| \leq 0.1$,

$$-0.1 \leq g - f \leq 0.1.$$

Integration over an area-6 region yields

$$-0.6 \leq \iint_R (g - f) \, dA \leq 0.6.$$

Consequently,

$$\boxed{18.4 \leq \iint_R g \, dA \leq 19.6}.$$

These endpoints are attainable by $g = f - 0.1$ and $g = f + 0.1$, so no tighter bounds follow from the supplied information.

Question 6

On the centrally symmetric rectangle

$$R = [-2, 2] \times [-1, 1],$$

consider $f(x, y) = 5 + x^3 + xy^2 + 2y$.

Tasks

1. Show that the nonconstant part cancels under central reflection.
2. Evaluate $\iint_R f \, dA$ and find the average value of f .
3. Explain precisely which symmetry of the region makes the argument valid.

Question 6 – Solution

Strategy. Pair each point (x, y) with its central reflection $(-x, -y)$ and average the two function values.

Step 1: Pairing Let $h(x, y) = x^3 + xy^2 + 2y$. Then

$$h(-x, -y) = -x^3 - xy^2 - 2y = -h(x, y).$$

Thus the contributions of h at every reflected pair cancel. Equivalently,

$$f(x, y) + f(-x, -y) = 10.$$

Step 2: Integral and average Central reflection maps R onto itself without changing area. Therefore the paired average of f is 5 everywhere, and

$$\iint_R f \, dA = 5 \operatorname{area}(R) = 5(4)(2) = \boxed{40}.$$

Dividing by the area 8 gives

$$\boxed{f_{\text{avg}} = 5}.$$

Step 3: Symmetry check The required property is

$$(x, y) \in R \iff (-x, -y) \in R.$$

Symmetry in only one coordinate would not automatically cancel the entire h : xy^2 is odd in x , while $2y$ is odd in y . Central symmetry reverses both coordinates at once and cancels all three terms.

Question 7

The plane $z = 6 + x - 2y$ lies above the rectangle

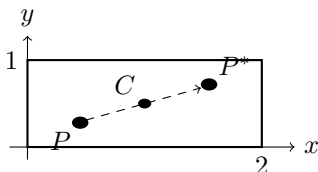
$$R = [0, 2] \times [0, 1].$$

Tasks

1. Pair each point of R with its reflection through the center $(1, 1/2)$ and compare their heights.
2. Use the pairing to find the volume under the plane without iterated integration.
3. Verify the result using the four corner heights.

Question 7 – Solution

Strategy. An affine function has complementary values at points reflected through a rectangle's center, so every reflected pair has the same average height.



Step 1: Reflected heights The reflection of $P = (x, y)$ through $C = (1, 1/2)$ is

$$P^* = (2 - x, 1 - y).$$

For $f(x, y) = 6 + x - 2y$,

$$\begin{aligned} f(P) + f(P^*) &= (6 + x - 2y) + (6 + (2 - x) - 2(1 - y)) \\ &= 12. \end{aligned}$$

Thus every pair has average height $6 = f(C)$.

Step 2: Volume The base area is 2, and the plane's minimum corner height is $4 > 0$, so the double integral is geometric volume:

$$V = \iint_R (6 + x - 2y) dA = 6(2) = 12.$$

Step 3: Corner check The corner heights are

$$f(0, 0) = 6, \quad f(2, 0) = 8, \quad f(0, 1) = 4, \quad f(2, 1) = 6.$$

Their average is $(6 + 8 + 4 + 6)/4 = 6$, agreeing with the center and pairing calculation. Multiplication by base area again gives 12.

Question 8

An affine function $f(x, y) = a + bx + cy$ is defined on the unit square. Its average value is 4 on the whole square, 5 on the right half $[1/2, 1] \times [0, 1]$, and 3 on the top half $[0, 1] \times [1/2, 1]$.

Tasks

1. Use center values of affine functions to obtain three equations for a, b, c .
2. Determine f uniquely.
3. Predict the integral of f over the lower-left quarter $[0, 1/2] \times [0, 1/2]$ and verify consistency.

Question 8 – Solution

Strategy. The average of an affine function over a rectangle equals its value at that rectangle's center; use the three given centers to reconstruct the coefficients.

Step 1: Average equations The relevant centers are $(1/2, 1/2)$, $(3/4, 1/2)$, and $(1/2, 3/4)$. Hence

$$\begin{aligned}a + \frac{b}{2} + \frac{c}{2} &= 4, \\a + \frac{3b}{4} + \frac{c}{2} &= 5, \\a + \frac{b}{2} + \frac{3c}{4} &= 3.\end{aligned}$$

The center-value rule follows from the same reflection pairing used for any affine graph over a rectangle.

Step 2: Reconstruct Subtracting the first equation from the second gives $b/4 = 1$, so $b = 4$. Subtracting the first from the third gives $c/4 = -1$, so $c = -4$. The first equation then gives $a = 4$. Therefore

$$\boxed{f(x, y) = 4 + 4x - 4y}.$$

Step 3: New prediction The lower-left quarter has center $(1/4, 1/4)$ and area $1/4$. Its average is

$$f\left(\frac{1}{4}, \frac{1}{4}\right) = 4 + 1 - 1 = 4,$$

so

$$\boxed{\iint_{[0, 1/2]^2} f \, dA = 4 \left(\frac{1}{4}\right) = 1}.$$

The equal x - and y -center coordinates cancel the opposite slopes, which checks the predicted average.

Question 9

Define the bounded function on the unit square by

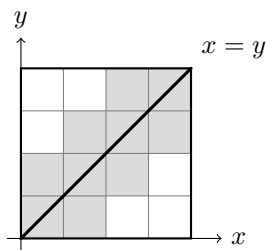
$$f(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Tasks

1. Explain why every lower sum is zero.
2. Use a uniform $n \times n$ grid to bound an upper sum by $3/n$.
3. Prove that f is Riemann integrable and find its double integral, despite its discontinuities.

Question 9 – Solution

Strategy. The diagonal has no area: trap every possible nonzero upper-sum contribution in a shrinking collection of grid squares.



Step 1: Lower sums Every subrectangle with positive area contains a point off the diagonal, where $f = 0$. Its infimum is therefore 0, so every lower sum equals 0.

Step 2: Upper-sum cover In an $n \times n$ grid, each square has area $1/n^2$. The n diagonal squares meet $x = y$. At each of the $n - 1$ interior diagonal grid vertices, two additional off-diagonal squares have closures meeting the diagonal. Thus at most

$$n + 2(n - 1) = 3n - 2$$

closed grid squares have supremum 1; all others have supremum 0. Consequently,

$$0 \leq U_n \leq \frac{3n - 2}{n^2} < \frac{3}{n}.$$

Step 3: Integrability Given $\varepsilon > 0$, choose $n > 3/\varepsilon$. Then

$$0 \leq U_n - L_n < \frac{3}{n} < \varepsilon.$$

The Darboux criterion proves integrability, and the integral is squeezed between zero lower sums and upper sums tending to zero:

$$\boxed{\iint_{[0,1]^2} f \, dA = 0}.$$

The function is discontinuous at every point of the diagonal, but that one-dimensional set contributes no area.

Question 10

A function f on the unit square satisfies the Lipschitz estimate

$$|f(P) - f(Q)| \leq 2\|P - Q\|$$

for all points P, Q . Let S_n be the midpoint Riemann sum on an $n \times n$ uniform grid.

Tasks

1. Prove the error bound $|\iint_R f \, dA - S_n| \leq \sqrt{2}/n$.
2. Find the least n certified by this bound to give error less than 0.01.
3. If $S_{142} = 2.731$, give a guaranteed interval containing the true integral.

Question 10 – Solution

Strategy. Control the variation within each cell by its maximum distance from the midpoint, then add the cellwise error bounds.

Step 1: One-cell error The Lipschitz estimate makes f continuous, hence integrable on the closed square. A grid square has side $1/n$, area $1/n^2$, and maximum distance from its midpoint to a corner

$$d_n = \frac{\sqrt{2}}{2n}.$$

If M_{ij} is its midpoint, the Lipschitz condition gives

$$|f(P) - f(M_{ij})| \leq 2d_n = \frac{\sqrt{2}}{n}$$

throughout that cell. Its integral differs from midpoint height times area by at most $\sqrt{2}/n^3$.

Step 2: Global error There are n^2 cells, so the triangle inequality yields

$$\left| \iint_R f \, dA - S_n \right| \leq n^2 \frac{\sqrt{2}}{n^3} = \frac{\sqrt{2}}{n}.$$

To guarantee strict error below 0.01, require $n > 100\sqrt{2} \approx 141.421$. The least integer is therefore n=142.

Step 3: Numerical enclosure For $n = 142$,

$$\frac{\sqrt{2}}{142} \approx 0.009959.$$

Thus

$$2.731 - \frac{\sqrt{2}}{142} \leq \iint_R f \, dA \leq 2.731 + \frac{\sqrt{2}}{142},$$

or approximately

$$2.72104 \leq \iint_R f \, dA \leq 2.74096.$$

The half-diagonal, rather than the full cell diagonal, is essential to the stated constant.