

Question 1

Find the constrained extrema of

$$f(x, y) = x + y$$

subject to

$$x^2 + y^2 = 1.$$

Tasks

1. Write and solve the Lagrange multiplier equations.
2. Determine the maximum and minimum values and their locations.
3. Interpret the multiplier condition as tangency of level curves.

Question 1 – Solution

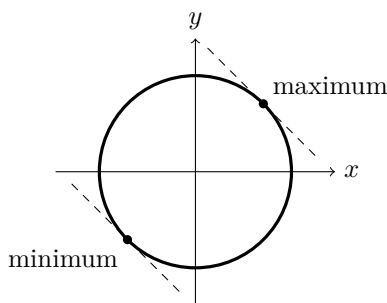
Strategy. Solve $\nabla f = \lambda \nabla g$ together with the circular constraint, then compare the finite candidate set.

Step 1: Lagrange system Let $g = x^2 + y^2$. Then

$$\langle 1, 1 \rangle = \lambda \langle 2x, 2y \rangle, \quad x^2 + y^2 = 1.$$

The first two equations imply $x = y$. Hence $2x^2 = 1$, so

$$(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \text{or} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$



Step 2: Values Direct evaluation gives

$$f_{\max} = \sqrt{2} \text{ at } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$f_{\min} = -\sqrt{2} \text{ at } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Step 3: Geometry and verification The level curves $x + y = c$ are parallel lines with normal ∇f . At an extreme one such line is tangent to the circle, so its normal is parallel to the radial normal ∇g . The circle is compact and the two candidates exhaust the multiplier system, completing the verification.

Question 2

Find the maximum and minimum of

$$f(x, y) = xy$$

subject to the ellipse

$$x^2 + 4y^2 = 4.$$

Tasks

1. Solve the multiplier equations, treating possible zero coordinates carefully.
2. List all constrained extrema and values.
3. Verify the result with a substitution that converts the constraint to a circle.

Question 2 – Solution

Strategy. Solve the component equations without dividing until zero-coordinate cases have been checked.

Step 1: Multiplier equations With $g = x^2 + 4y^2$,

$$\langle y, x \rangle = \lambda \langle 2x, 8y \rangle, \quad x^2 + 4y^2 = 4.$$

If $x = 0$, the first multiplier equation forces $y = 0$, violating the constraint; similarly $y = 0$ is impossible. Thus $xy \neq 0$, and

$$y = 2\lambda x, \quad x = 8\lambda y.$$

Consequently $16\lambda^2 = 1$ and $y = \pm x/2$. The constraint then gives $2x^2 = 4$, so $x = \pm\sqrt{2}$.

Step 2: Candidates and values Equal signs for x and y give $xy = 1$; opposite signs give $xy = -1$. Therefore

$$f_{\max} = 1 \text{ at } (\sqrt{2}, \sqrt{2}/2), (-\sqrt{2}, -\sqrt{2}/2),$$

$$f_{\min} = -1 \text{ at } (\sqrt{2}, -\sqrt{2}/2), (-\sqrt{2}, \sqrt{2}/2).$$

Step 3: Verification Let $u = x$ and $v = 2y$. Then $u^2 + v^2 = 4$ and $f = uv/2$. Since

$$|uv| \leq \frac{u^2 + v^2}{2} = 2,$$

we have $|f| \leq 1$, with equality precisely when $|u| = |v|$. This reproduces all four points.

Question 3

Find the points on the parabola

$$y = x^2$$

closest to $A = (0, 2)$.

Tasks

1. Formulate the squared-distance objective and the constraint.
2. Solve the Lagrange system, including all branches.
3. Compare candidates and justify that the reported minimum is absolute despite the noncompact constraint.

Question 3 – Solution

Strategy. Minimize squared distance, which has the same minimizers as distance but simpler derivatives.

Step 1: Set up Let

$$f(x, y) = x^2 + (y - 2)^2, \quad g(x, y) = y - x^2.$$

The system $\nabla f = \lambda \nabla g$ is

$$2x = -2\lambda x, \quad 2(y - 2) = \lambda, \quad y = x^2.$$

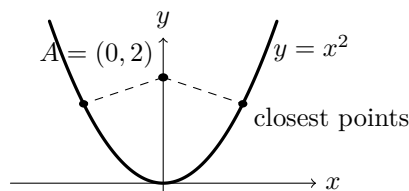
Step 2: Solve branches If $x = 0$, then $y = 0$, producing the candidate $(0, 0)$ with squared distance 4.

If $x \neq 0$, the first equation gives $\lambda = -1$. The second gives $2(y - 2) = -1$, so $y = 3/2$. Hence

$$x = \pm \sqrt{\frac{3}{2}}.$$

At either point the squared distance is

$$\frac{3}{2} + \left(-\frac{1}{2}\right)^2 = \boxed{\frac{7}{4}}.$$



Step 3: Conclusion Comparing candidates gives the two closest points

$$\boxed{\left(\pm \sqrt{\frac{3}{2}}, \frac{3}{2}\right)},$$

each at distance $\boxed{\sqrt{7}/2}$ from A . Along the parabola, $f(x, x^2) = x^2 + (x^2 - 2)^2 \rightarrow \infty$ as $|x| \rightarrow \infty$. Therefore no minimizing sequence can escape to infinity, and the smaller candidate value is the absolute minimum.

Question 4

A rectangular box has positive side lengths x, y, z and fixed sum of its three side lengths

$$x + y + z = 12.$$

Find the dimensions giving the largest volume $V = xyz$.

Tasks

1. Solve the three-variable Lagrange multiplier system.
2. Find the maximum volume.
3. Verify global maximality by checking the boundary of the closed simplex and using an inequality.

Question 4 – Solution

Strategy. The multiplier equations force equal positive side lengths; then compare with the zero-volume boundary and verify by AM–GM.

Step 1: Lagrange system For $g = x + y + z$,

$$\nabla V = \langle yz, xz, xy \rangle = \lambda \langle 1, 1, 1 \rangle.$$

Because $x, y, z > 0$, equality of the three components gives

$$yz = xz = xy,$$

and division by positive factors yields $x = y = z$. The constraint then gives

$$\boxed{x = y = z = 4}.$$

Step 2: Volume

$$\boxed{V_{\max} = 4 \cdot 4 \cdot 4 = 64}.$$

Step 3: Global verification Allowing $x, y, z \geq 0$ closes the feasible simplex. Its boundary has at least one zero side, so $V = 0$ there. In the positive interior, AM–GM gives

$$\frac{x + y + z}{3} \geq (xyz)^{1/3}.$$

Since the left side is 4,

$$xyz \leq 64,$$

with equality only when $x = y = z$. Thus the Lagrange candidate is the unique positive absolute maximizer.

Question 5

The sphere

$$x^2 + y^2 + z^2 = 9$$

meets the plane

$$x + y + z = 3.$$

Find the highest and lowest points of their intersection curve.

Tasks

1. Use two Lagrange multipliers to extremize the height $f(x, y, z) = z$.
2. Solve the complete system and identify both points.
3. Verify the result geometrically or by direct substitution.

Question 5 – Solution

Strategy. At a constrained extreme with two regular constraints, ∇f lies in the span of both constraint gradients.

Step 1: Two-multiplier system Let

$$g = x^2 + y^2 + z^2, \quad h = x + y + z.$$

Then

$$\langle 0, 0, 1 \rangle = \lambda \langle 2x, 2y, 2z \rangle + \mu \langle 1, 1, 1 \rangle.$$

The first two component equations imply $2\lambda(x - y) = 0$. If $\lambda = 0$, then the first equation gives $\mu = 0$, contradicting the third. Hence $x = y = a$.

Step 2: Enforce both constraints The plane gives $z = 3 - 2a$. Substitution into the sphere gives

$$2a^2 + (3 - 2a)^2 = 9,$$

or

$$6a(a - 2) = 0.$$

Thus $a = 0$, $z = 3$, or $a = 2$, $z = -1$. Therefore

$$\boxed{\text{highest point } (0, 0, 3)}, \quad \boxed{\text{lowest point } (2, 2, -1)}.$$

Step 3: Verification Both points satisfy the sphere and plane equations. Their heights are 3 and -1 . The intersection is a closed circle, hence compact; the two candidates exhaust the multiplier equations, so these are the absolute height extrema.

Question 6

Consider maximizing and minimizing

$$f(x, y) = x$$

subject to the constraint

$$g(x, y) = x^2 + y^2 = 0.$$

Tasks

1. Determine the feasible set and its constrained extrema directly.
2. Show that no multiplier λ satisfies $\nabla f = \lambda \nabla g$ at the feasible point.
3. Explain which constraint-qualification hypothesis fails and why this does not contradict the method.

Question 6 – Solution

Strategy. Inspect the feasible set before applying the multiplier equation; the constraint has a singular gradient at its only point.

Step 1: Direct analysis The equation $x^2 + y^2 = 0$ over the reals forces

$$x = 0, \quad y = 0.$$

Thus the feasible set is the singleton $\{(0, 0)\}$. Its only function value is 0, so

$$\boxed{f_{\min} = f_{\max} = 0 \text{ at } (0, 0)}.$$

Step 2: Multiplier failure

$$\nabla f(0, 0) = \langle 1, 0 \rangle, \quad \nabla g(0, 0) = \langle 0, 0 \rangle.$$

The equation

$$\langle 1, 0 \rangle = \lambda \langle 0, 0 \rangle$$

has no solution for any real λ .

Step 3: Interpretation The standard Lagrange multiplier theorem assumes $\nabla g \neq \mathbf{0}$ at the constrained extremum. That regularity condition fails here, so the theorem makes no promise. The example shows that singular feasible points must be checked separately; it does not invalidate the multiplier method at regular constraint points.

Question 7

Find the constrained extrema of

$$f(x, y) = x^2 + y^2$$

subject to

$$x^2 + xy + y^2 = 3.$$

Tasks

1. Solve the Lagrange equations without assuming x or y is nonzero.
2. Determine all maximizing and minimizing points and values.
3. Verify the result by rewriting the constraint in rotated directions $y = x$ and $y = -x$.

Question 7 – Solution

Strategy. Eliminate the multiplier by combining the two component equations; the resulting factorization identifies the principal directions.

Step 1: Multiplier equations

$$\langle 2x, 2y \rangle = \lambda \langle 2x + y, x + 2y \rangle, \quad x^2 + xy + y^2 = 3.$$

Multiply the first component equation by y , the second by x , and subtract. The left sides cancel, leaving

$$\lambda(y^2 - x^2) = 0.$$

The constraint rules out $(0, 0)$, and $\lambda = 0$ would force $x = y = 0$. Hence

$$y = \pm x.$$

Step 2: Candidate values If $y = x$, the constraint is $3x^2 = 3$, so $(x, y) = (1, 1)$ or $(-1, -1)$ and $f = 2$.

If $y = -x$, the constraint is $x^2 = 3$, giving $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ with $f = 6$. Therefore

$$f_{\min} = 2 \text{ at } (1, 1), (-1, -1),$$

$$f_{\max} = 6 \text{ at } (\sqrt{3}, -\sqrt{3}), (-\sqrt{3}, \sqrt{3}).$$

Step 3: Verification The constraint ellipse is compact. Along its $y = x$ principal direction its radius squared is 2, while along $y = -x$ its radius squared is 6. These are exactly the values of $x^2 + y^2$, confirming the classifications and completeness.

Question 8

For the family

$$f_a(x, y) = x^2 + ay^2,$$

the point $P = (1, 2)$ is a constrained extremum on the line $x + y = 3$.

Tasks

1. Determine the parameter a using the Lagrange condition.
2. Classify the constrained extremum.
3. Verify the classification by substituting the constraint into the resulting function.

Question 8 – Solution

Strategy. Parallel gradients at the specified regular point determine the parameter; a one-variable restriction then establishes the classification.

Step 1: Recover a For $g = x + y$,

$$\nabla f_a(1, 2) = \langle 2, 4a \rangle, \quad \nabla g = \langle 1, 1 \rangle.$$

The equation $\langle 2, 4a \rangle = \lambda \langle 1, 1 \rangle$ gives $\lambda = 2$ and $4a = 2$. Hence

$$\boxed{a = \frac{1}{2}}.$$

Step 2: Restrict to the line Set $y = 3 - x$. Then

$$f_{1/2}(x, 3 - x) = x^2 + \frac{1}{2}(3 - x)^2.$$

Completing the square,

$$x^2 + \frac{1}{2}(3 - x)^2 = \frac{3}{2}(x - 1)^2 + 3.$$

Therefore the constrained value is minimized uniquely at $x = 1$, $y = 2$.

Step 3: Result and verification

$$\boxed{P = (1, 2) \text{ is a strict constrained minimum}}, \quad \boxed{f_{1/2}(1, 2) = 3}.$$

The restricted quadratic tends to infinity as $|x| \rightarrow \infty$, so the minimum is also absolute along the entire constraint line; no constrained maximum exists.

Question 9

A firm must choose positive inputs x, y satisfying

$$xy = 100.$$

Its cost is

$$C(x, y) = 3x + 2y.$$

Tasks

1. Use Lagrange multipliers to find the cost-minimizing inputs.
2. Compute the minimum cost and interpret the multiplier balance.
3. Verify global minimality by substitution or an inequality.

Question 9 – Solution

Strategy. The multiplier equations balance marginal costs against the constraint normal; then a one-variable check rules out other behavior.

Step 1: Multiplier equations With $g = xy$,

$$\langle 3, 2 \rangle = \lambda \langle y, x \rangle, \quad xy = 100.$$

Thus $3 = \lambda y$ and $2 = \lambda x$. Eliminating λ gives

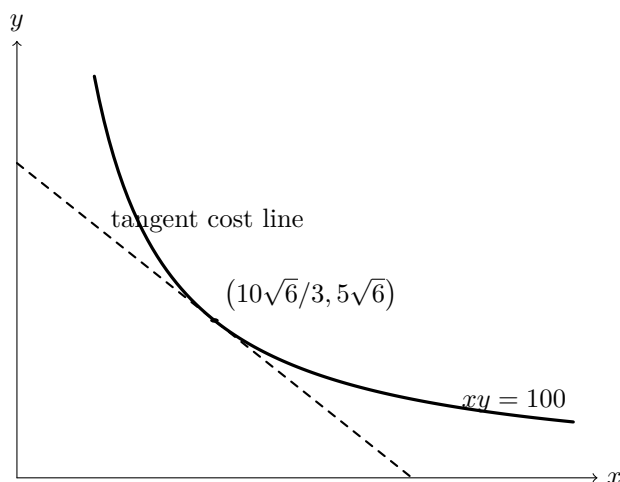
$$3x = 2y, \quad y = \frac{3}{2}x.$$

The positive constraint then yields

$$x^2 = \frac{200}{3},$$

so

$$x = \frac{10\sqrt{6}}{3}, \quad y = 5\sqrt{6}.$$



Step 2: Cost and interpretation

$$C_{\min} = 3 \left(\frac{10\sqrt{6}}{3} \right) + 2(5\sqrt{6}) = 20\sqrt{6}.$$

The relation $3/y = 2/x = \lambda$ says the cost-gradient components are proportional to the production-constraint gradient components.

Step 3: Global verification Using $y = 100/x$ for $x > 0$,

$$C(x) = 3x + \frac{200}{x} \longrightarrow \infty$$

as $x \rightarrow 0^+$ or $x \rightarrow \infty$. Its only critical point is the one found above, and $C''(x) = 400/x^3 > 0$. Hence it is the unique absolute constrained minimum.

Question 10

Find the absolute extrema of

$$f(x, y, z) = xyz$$

on the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6.$$

Tasks

1. Solve the Lagrange equations for candidates with nonzero product.
2. Treat zero-coordinate candidates and compare all possible values.
3. List every maximizing and minimizing point and verify the sign pattern.

Question 10 – Solution

Strategy. Multiplying each component equation by its corresponding coordinate reveals fixed ratios among x^2, y^2, z^2 .

Step 1: Nonzero-product candidates The multiplier system is

$$yz = 2\lambda x, \quad xz = 4\lambda y, \quad xy = 6\lambda z.$$

When $xyz \neq 0$, multiply these equations by x, y, z , respectively:

$$xyz = 2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2.$$

Thus

$$2x^2 = 4y^2 = 6z^2 = K.$$

Substitution into the constraint gives

$$\frac{K}{2} + 2\frac{K}{4} + 3\frac{K}{6} = \frac{3K}{2} = 6,$$

so $K = 4$. Therefore

$$x^2 = 2, \quad y^2 = 1, \quad z^2 = \frac{2}{3}.$$

Step 2: Values and zero cases At these eight sign choices,

$$|xyz| = \sqrt{2} \cdot 1 \cdot \sqrt{\frac{2}{3}} = \frac{2}{\sqrt{3}}.$$

Any feasible candidate with a zero coordinate has $f = 0$, which lies strictly between the positive and negative nonzero values.

Step 3: All extrema The ellipsoid is compact, so comparison is decisive:

$$f_{\max} = \frac{2}{\sqrt{3}}$$

at the four points $(\pm\sqrt{2}, \pm 1, \pm\sqrt{2/3})$ having an even number of minus signs, and

$$f_{\min} = -\frac{2}{\sqrt{3}}$$

at the four such points having an odd number of minus signs. The parity rule follows because the product is positive for an even number of negative factors and negative for an odd number.