

Question 1

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 2x - 4y$$

on the rectangle $R = [0, 3] \times [0, 4]$.

Tasks

1. Find all interior critical points.
2. Analyze all four boundary segments, including their endpoints.
3. Compare the complete candidate list and report every absolute extremum.

Question 1 – Solution

Strategy. On a closed, bounded rectangle, compare values at interior critical points, one-variable boundary critical points, and corners.

Step 1: Interior

$$f_x = 2x - 2, \quad f_y = 2y - 4.$$

The sole interior critical point is $(1, 2)$, where

$$f(1, 2) = 1 + 4 - 2 - 8 = \boxed{-5}.$$

Step 2: Boundaries On $x = 0$ and $x = 3$,

$$f(0, y) = y^2 - 4y, \quad f(3, y) = y^2 - 4y + 3.$$

Both have a boundary critical value at $y = 2$, giving -4 and -1 . On $y = 0$ and $y = 4$,

$$f(x, 0) = f(x, 4) = x^2 - 2x,$$

whose critical point $x = 1$ gives -1 . The four corner values are

$$f(0, 0) = 0, \quad f(3, 0) = 3, \quad f(0, 4) = 0, \quad f(3, 4) = 3.$$

Step 3: Comparison The smallest candidate value is -5 and the largest is 3 . Hence

$$\boxed{f_{\min} = -5 \text{ at } (1, 2)},$$

$$\boxed{f_{\max} = 3 \text{ at } (3, 0) \text{ and } (3, 4)}.$$

Completing the square, $f = (x - 1)^2 + (y - 2)^2 - 5$, independently confirms the comparison.

Question 2

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 2x$$

on the closed unit disk $D : x^2 + y^2 \leq 1$.

Tasks

1. Check for interior critical points.
2. Parameterize the circular boundary and find all boundary candidates.
3. Compare the candidates and verify the result geometrically by completing the square.

Question 2 – Solution

Strategy. Treat the interior and the circular boundary separately; on the boundary the radial term is constant.

Step 1: Interior

$$f_x = 2x - 2, \quad f_y = 2y.$$

The only solution is $(1, 0)$, which lies on the boundary rather than in the interior. Thus there are no interior critical points.

Step 2: Boundary Set $x = \cos \theta$, $y = \sin \theta$. Then

$$f(\cos \theta, \sin \theta) = 1 - 2 \cos \theta.$$

The smallest value occurs when $\cos \theta = 1$ and the largest when $\cos \theta = -1$. Therefore

$$\boxed{f_{\min} = -1 \text{ at } (1, 0)}, \quad \boxed{f_{\max} = 3 \text{ at } (-1, 0)}.$$

Step 3: Geometric verification

$$f(x, y) = (x - 1)^2 + y^2 - 1.$$

Thus $f + 1$ is squared distance from (x, y) to $(1, 0)$. Within the disk, that distance is smallest at $(1, 0)$ and largest at the diametrically opposite point $(-1, 0)$. This confirms both locations and values.

Question 3

On the closed triangle

$$T = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 4\},$$

find the absolute extrema of

$$f(x, y) = xy(4 - x - y).$$

Tasks

1. Find every interior critical point.
2. Analyze all three edges and vertices.
3. Compare the candidates and interpret why the minimum occurs along an entire boundary.

Question 3 – Solution

Strategy. The compact triangle guarantees extrema; compare the interior stationary value with the three edge restrictions.

Step 1: Interior

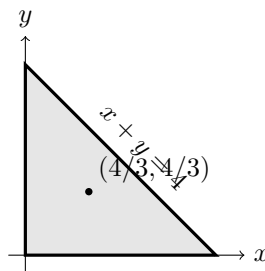
$$f_x = y(4 - 2x - y), \quad f_y = x(4 - x - 2y).$$

In the interior $x, y > 0$, so

$$2x + y = 4, \quad x + 2y = 4.$$

Thus $x = y = 4/3$, and

$$f\left(\frac{4}{3}, \frac{4}{3}\right) = \frac{4}{3} \frac{4}{3} \frac{4}{3} = \boxed{\frac{64}{27}}.$$



Step 2: Boundary On $x = 0$ or $y = 0$, one factor vanishes. On $x + y = 4$, the third factor vanishes. Hence

$$f = 0 \quad \text{on all three edges, including the vertices.}$$

Step 3: Comparison Inside T , all three factors x , y , and $4 - x - y$ are positive, so $f > 0$. Therefore

$$\boxed{f_{\min} = 0 \text{ at every boundary point of } T},$$

$$\boxed{f_{\max} = \frac{64}{27} \text{ at } (4/3, 4/3)}.$$

The candidate list is complete because the polynomial is continuous and T is closed and bounded.

Question 4

Find the absolute extrema of

$$f(x, y) = x + 2y$$

on and inside the ellipse

$$x^2 + 4y^2 \leq 4.$$

Do not use Lagrange multipliers.

Tasks

1. Transform the ellipse to a disk with a change of variables.
2. Use an inequality to derive sharp upper and lower bounds.
3. Find all equality points and verify they lie on the ellipse.

Question 4 – Solution

Strategy. Let $u = x$ and $v = 2y$; then the region is a radius-2 disk and the objective is $u + v$.

Step 1: Transform Set

$$u = x, \quad v = 2y.$$

The constraint becomes $u^2 + v^2 \leq 4$, while

$$f = u + v = \langle u, v \rangle \cdot \langle 1, 1 \rangle.$$

Step 2: Sharp bounds By Cauchy–Schwarz,

$$|u + v| \leq \sqrt{u^2 + v^2} \sqrt{2} \leq 2\sqrt{2}.$$

Thus $-2\sqrt{2} \leq f \leq 2\sqrt{2}$.

Equality on the upper side requires $\langle u, v \rangle$ to point with $\langle 1, 1 \rangle$ and have length 2, giving

$$(u, v) = (\sqrt{2}, \sqrt{2}).$$

Lower equality gives its negative.

Step 3: Return to (x, y) Because $x = u$ and $y = v/2$,

$$f_{\max} = 2\sqrt{2} \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2} \right),$$

$$f_{\min} = -2\sqrt{2} \text{ at } \left(-\sqrt{2}, -\frac{\sqrt{2}}{2} \right).$$

For either point, $x^2 + 4y^2 = 2 + 2 = 4$, verifying boundary membership and sharpness.

Question 5

Find the absolute extrema of

$$f(x, y) = x^2 + y^2 - 3x$$

on the closed annulus

$$1 \leq x^2 + y^2 \leq 4.$$

Tasks

1. Find all interior critical points that belong to the annulus.
2. Analyze both circular boundary components.
3. Compare every candidate and identify the absolute extrema.

Question 5 – Solution

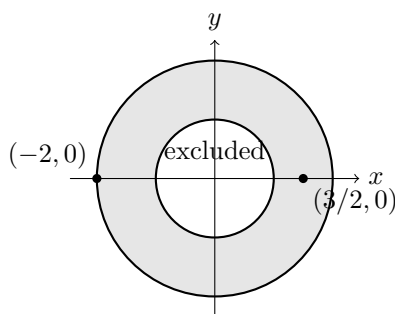
Strategy. Check the annulus interior, then write each boundary circle as $x = r \cos \theta$, $y = r \sin \theta$ with fixed r .

Step 1: Interior

$$f_x = 2x - 3, \quad f_y = 2y.$$

The critical point is $(3/2, 0)$, whose radius $3/2$ lies strictly between 1 and 2. Its value is

$$f(3/2, 0) = \frac{9}{4} - \frac{9}{2} = \boxed{-\frac{9}{4}}.$$



Step 2: Circular boundaries On a circle of radius r ,

$$f = r^2 - 3r \cos \theta.$$

For $r = 1$, its range is $[-2, 4]$. For $r = 2$, its range is $[-2, 10]$. The value 10 occurs on the outer circle when $\cos \theta = -1$, at $(-2, 0)$.

Step 3: Comparison Comparing $-9/4$, the boundary minimum -2 , and the boundary maximum 10 gives

$$\boxed{f_{\min} = -\frac{9}{4} \text{ at } (3/2, 0)}, \quad \boxed{f_{\max} = 10 \text{ at } (-2, 0)}.$$

Both boundary components were checked, so the hole in the region causes no missing candidates.

Question 6

Let

$$D = \{(x, y) : x^2 + y^2 < 1\}$$

be the open unit disk, and let $f(x, y) = x + y$.

Tasks

1. Determine whether f has an absolute maximum or minimum on D .
2. Find its supremum and infimum and give sequences approaching them.
3. Explain exactly which hypothesis of the Extreme Value Theorem is absent.

Question 6 – Solution

Strategy. Bound the linear function with Cauchy–Schwarz, then determine whether equality points belong to the open disk.

Step 1: Bounds For $(x, y) \in D$,

$$|x + y| \leq \sqrt{x^2 + y^2} \sqrt{2} < \sqrt{2}.$$

Therefore every function value lies strictly between $-\sqrt{2}$ and $\sqrt{2}$.

Step 2: Approaching the bounds For $0 < r < 1$, the point

$$\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}} \right)$$

lies in D and gives $f = r\sqrt{2} \rightarrow \sqrt{2}$ as $r \rightarrow 1^-$. Its negative gives values tending to $-\sqrt{2}$. Thus

$$\boxed{\sup_D f = \sqrt{2}}, \quad \boxed{\inf_D f = -\sqrt{2}}.$$

Neither value is attained, so f has no absolute maximum or minimum on D .

Step 3: The theorem The function is continuous and the disk is bounded, but D is not closed: its boundary circle is omitted. Hence the domain is not compact, and the Extreme Value Theorem does not guarantee attained extrema. The missing equality points are precisely

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right).$$

Question 7

On the unbounded domain $\mathbb{R}^2$, consider

$$f(x, y) = x^2 + y^2 + \frac{1}{1 + x^2 + y^2}.$$

Tasks

1. Determine all absolute minima and their value.
2. Decide whether an absolute maximum exists.
3. Explain why this example shows that compactness is sufficient, but not necessary, for an absolute minimum to exist.

Question 7 – Solution

Strategy. Reduce the function to the radial variable $s = x^2 + y^2 \geq 0$ and study a one-variable function on an unbounded interval.

Step 1: Radial reduction Let

$$g(s) = s + \frac{1}{1+s}, \quad s \geq 0.$$

Then

$$g'(s) = 1 - \frac{1}{(1+s)^2}.$$

At $s = 0$, $g'(0) = 0$, and for every $s > 0$, $(1+s)^2 > 1$, so $g'(s) > 0$. Therefore g increases strictly away from $s = 0$.

It follows that

$$\boxed{f_{\min} = 1 \text{ at } (0,0)}.$$

Step 2: No maximum As $x^2 + y^2 = s \rightarrow \infty$,

$$g(s) = s + \frac{1}{1+s} \rightarrow \infty.$$

Thus f is unbounded above and has $\boxed{\text{no absolute maximum}}$.

Step 3: Interpretation The domain $\mathbb{R}^2$ is closed but not bounded, so the Extreme Value Theorem does not apply. Nevertheless, direct analysis proves that an absolute minimum is attained. Compactness guarantees both extrema for every continuous function, but a particular continuous function may still attain one or both extrema on a noncompact domain.

Question 8

For a real parameter a , define

$$f_a(x, y) = (x - a)^2 + y^2$$

on the square $S = [-1, 1] \times [-1, 1]$.

Tasks

1. Determine the absolute minimum value and all minimizing points for every a .
2. Determine the absolute maximum value and all maximizing points for every a .
3. Treat the tie case $a = 0$ explicitly and interpret the answer as squared distance.

Question 8 – Solution

Strategy. The function is squared distance from (x, y) to $(a, 0)$; minimize and maximize the two nonnegative coordinate contributions independently.

Step 1: Minimum The y^2 term is minimized uniquely at $y = 0$. The closest $x \in [-1, 1]$ to a is the clipped value

$$x_* = \begin{cases} -1, & a < -1, \\ a, & -1 \leq a \leq 1, \\ 1, & a > 1. \end{cases}$$

Hence the unique minimizing point is $(x_*, 0)$, and

$$f_{\min} = (\max\{|a| - 1, 0\})^2.$$

Step 2: Maximum The largest possible y^2 is 1, attained at $y = \pm 1$. The endpoint of $[-1, 1]$ farthest from a is at distance $|a| + 1$. Therefore

$$f_{\max} = (|a| + 1)^2 + 1.$$

If $a > 0$, the maximizers are $(-1, \pm 1)$; if $a < 0$, they are $(1, \pm 1)$.

Step 3: Tie case When $a = 0$, both x -endpoints are equally far away, so all four corners $(\pm 1, \pm 1)$ maximize f_0 with value 2. The point $(0, 0)$ uniquely minimizes it with value 0. This agrees exactly with the squared-distance interpretation.

Question 9

Let R be the closed region between

$$y = x^2 \quad \text{and} \quad y = 2 - x.$$

Find the absolute extrema of

$$f(x, y) = x^2 + y$$

on R .

Tasks

1. Find the boundary intersection points and describe the compact region.
2. Check the interior and each curved boundary piece.
3. Compare all candidates and verify the result from the geometry of the region.

Question 9 – Solution

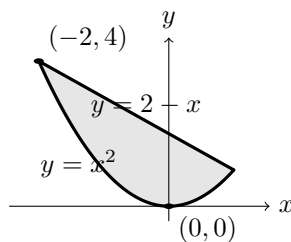
Strategy. Determine the finite interval where the line lies above the parabola, then reduce each boundary to one variable.

Step 1: Region The boundaries meet when

$$x^2 = 2 - x,$$

so $(x + 2)(x - 1) = 0$. The intersection points are $(-2, 4)$ and $(1, 1)$, and

$$-2 \leq x \leq 1, \quad x^2 \leq y \leq 2 - x.$$



Step 2: Candidate search Since $f_y = 1$, there are no interior critical points. On the lower boundary $y = x^2$,

$$f = 2x^2,$$

whose minimum is 0 at $x = 0$ and whose largest endpoint value is 8 at $x = -2$. On the upper boundary $y = 2 - x$,

$$f = x^2 - x + 2 = \left(x - \frac{1}{2}\right)^2 + \frac{7}{4}.$$

Its boundary minimum is $7/4$ at $x = 1/2$; its endpoint values are 8 and 2.

Step 3: Comparison Therefore

$$\boxed{f_{\min} = 0 \text{ at } (0, 0)}, \quad \boxed{f_{\max} = 8 \text{ at } (-2, 4)}.$$

Because $f = x^2 + y$ increases with y , it is also geometrically consistent that vertical comparisons reduce to the lower curve for the minimum and upper curve for the maximum.

Question 10

Find the absolute extrema of the nondifferentiable function

$$f(x, y) = |x| + y^2$$

on the closed unit disk $x^2 + y^2 \leq 1$.

Tasks

1. Analyze the interior, including the crease $x = 0$ where f_x does not exist.
2. Reduce the circular boundary problem to a single variable $u = |x|$.
3. Find and verify all absolute minimizing and maximizing points.

Question 10 – Solution

Strategy. A complete candidate search must include nondifferentiable points. On the circle, replace y^2 by $1 - x^2$.

Step 1: Interior and crease Where $x \neq 0$,

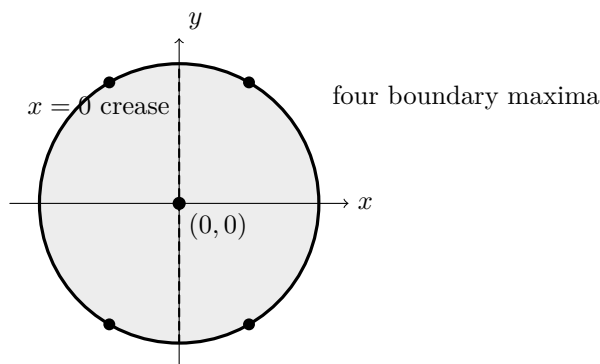
$$f_x = \operatorname{sgn}(x), \quad f_y = 2y,$$

so there are no differentiable interior critical points. On the crease $x = 0$,

$$f(0, y) = y^2.$$

Its smallest value is 0 at the origin. Since $|x| \geq 0$ and $y^2 \geq 0$ everywhere,

$$\boxed{f_{\min} = 0 \text{ at } (0, 0)}.$$



Step 2: Circular boundary Let $u = |x| \in [0, 1]$. On $x^2 + y^2 = 1$,

$$f = u + 1 - u^2 = \frac{5}{4} - \left(u - \frac{1}{2}\right)^2.$$

Thus the largest boundary value is $5/4$, attained when $|x| = 1/2$. Then

$$y^2 = 1 - \frac{1}{4} = \frac{3}{4},$$

so $y = \pm\sqrt{3}/2$.

Step 3: Final comparison For a fixed x , increasing y^2 increases f , so an absolute maximum must lie on the circle; the boundary calculation is therefore decisive. Hence

$$\boxed{f_{\max} = \frac{5}{4} \text{ at } \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)},$$

where the two signs are independent, giving four points. Direct substitution returns $1/2 + 3/4 = 5/4$ at each.