

Question 1

For

$$f(x, y) = x^2 + 2y^2 - 4x + 8y + 1,$$

analyze all relative extrema.

Tasks

1. Find every critical point.
2. Classify each point with the second derivative test.
3. Complete the square to verify the classification and find the extremal value.

Question 1 – Solution

Strategy. Solve the first-derivative equations, classify with the Hessian determinant, and then use an exact algebraic form as an independent check.

Step 1: Critical point

$$f_x = 2x - 4, \quad f_y = 4y + 8.$$

Both vanish only when

$$\boxed{(x, y) = (2, -2)}.$$

Step 2: Second derivative test

$$f_{xx} = 2, \quad f_{yy} = 4, \quad f_{xy} = 0.$$

The Hessian determinant is

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 8 > 0.$$

Since $f_{xx} > 0$, the point is a strict relative minimum. Its value is

$$f(2, -2) = \boxed{-11}.$$

Step 3: Algebraic verification Completing squares gives

$$f(x, y) = (x - 2)^2 + 2(y + 2)^2 - 11.$$

Both squared terms are nonnegative and vanish simultaneously only at $(2, -2)$. This directly confirms the strict local minimum and rules out any other critical point or relative extremum.

Question 2

For

$$f(x, y) = x^3 + y^3 - 3xy,$$

find and classify every critical point.

Tasks

1. Solve the nonlinear critical-point equations without losing solutions.
2. Apply the second derivative test at each point.
3. Verify the saddle classification by displaying nearby values of opposite signs relative to the critical value.

Question 2 – Solution

Strategy. Use the coupled equations to reduce the possible coordinates, then apply the Hessian determinant separately at each solution.

Step 1: Critical points

$$f_x = 3x^2 - 3y, \quad f_y = 3y^2 - 3x.$$

Thus $y = x^2$ and $x = y^2$. Substitution gives

$$x = x^4, \quad x(x^3 - 1) = 0.$$

The real possibilities are $x = 0$ and $x = 1$, giving

$$\boxed{(0, 0) \quad \text{and} \quad (1, 1)}.$$

Step 2: Hessian test

$$f_{xx} = 6x, \quad f_{yy} = 6y, \quad f_{xy} = -3,$$

so $D = 36xy - 9$. At $(0, 0)$, $D = -9 < 0$, hence it is a saddle point. At $(1, 1)$, $D = 27 > 0$ and $f_{xx} = 6 > 0$, so it is a strict relative minimum with

$$f(1, 1) = \boxed{-1}.$$

Step 3: Saddle verification At the origin the critical value is 0. Along $y = 0$,

$$f(x, 0) = x^3,$$

which is positive for small $x > 0$ and negative for small $x < 0$. Thus every neighborhood contains values above and below 0, independently verifying the saddle classification.

Question 3

Analyze the origin for

$$f(x, y) = x^2 + 4xy + y^2.$$

Tasks

1. Verify that the origin is the only critical point.
2. Classify it using the second derivative test.
3. Give two explicit paths that make the saddle geometry visible.

Question 3 – Solution

Strategy. The quadratic form itself records the local geometry; the Hessian test and carefully chosen diagonal paths should agree.

Step 1: Critical point

$$f_x = 2x + 4y, \quad f_y = 4x + 2y.$$

Solving $x + 2y = 0$ and $2x + y = 0$ gives $x = y = 0$. Therefore the origin is the unique critical point.

Step 2: Hessian test

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 4.$$

Hence

$$D = (2)(2) - 4^2 = -12 < 0.$$

Therefore

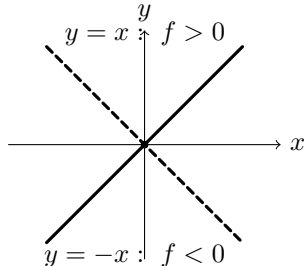
$(0, 0)$ is a saddle point.

Step 3: Path verification Along $y = x$,

$$f(x, x) = x^2 + 4x^2 + x^2 = 6x^2 > 0$$

for $x \neq 0$. Along $y = -x$,

$$f(x, -x) = x^2 - 4x^2 + x^2 = -2x^2 < 0.$$



Both paths approach the origin, whose value is zero, so arbitrarily nearby values occur on both sides of the critical value.

Question 4

Consider the “monkey saddle”

$$f(x, y) = x^3 - 3xy^2.$$

Tasks

1. Find all critical points.
2. Show precisely why the second derivative test is inconclusive there.
3. Classify the point using both simple paths and the polar form of the function.

Question 4 – Solution

Strategy. When the Hessian degenerates, return to the definition and compare the function along paths through the critical point.

Step 1: Critical point

$$f_x = 3x^2 - 3y^2, \quad f_y = -6xy.$$

The second equation requires $x = 0$ or $y = 0$. In either case the first equation then forces the other coordinate to be zero. Thus the only critical point is $(0, 0)$.

Step 2: Failed Hessian test

$$f_{xx} = 6x, \quad f_{yy} = -6x, \quad f_{xy} = -6y.$$

All three second derivatives vanish at the origin, so

$$D(0, 0) = 0.$$

The test gives no classification.

Step 3: Direct classification

Along $y = 0$,

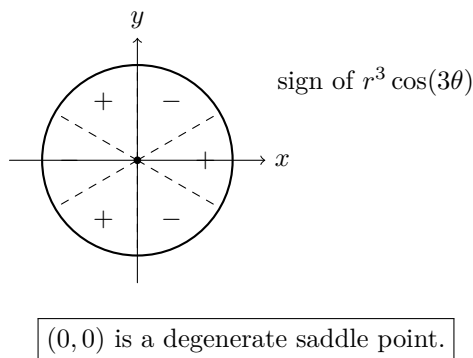
$$f(x, 0) = x^3,$$

which takes both signs arbitrarily close to zero. Hence the origin is a saddle.

For a structural check, let $x = r \cos \theta$ and $y = r \sin \theta$. Then

$$f = r^3 (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos(3\theta).$$

The factor $\cos(3\theta)$ alternates sign in six angular sectors. Therefore



Question 5

Let

$$f(x, y) = (x^2 + y^2)e^{-(x^2 + y^2)}.$$

Tasks

1. Find the complete critical set.
2. Classify the origin and every point on the critical circle as relative extrema.
3. Explain why the circular extrema are not strict and why the ordinary Hessian test degenerates there.

Question 5 – Solution

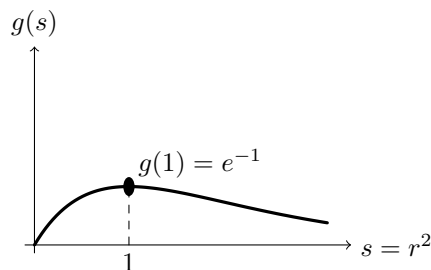
Strategy. Set $s = x^2 + y^2$ and analyze the one-variable radial profile $g(s) = se^{-s}$.

Step 1: Critical set Since

$$f_x = 2xe^{-s}(1-s), \quad f_y = 2ye^{-s}(1-s),$$

both derivatives vanish when $(x, y) = (0, 0)$ or when $s = 1$. Thus

$$\text{critical set} = \{(0, 0)\} \cup \{x^2 + y^2 = 1\}.$$



Step 2: Radial classification For $s \geq 0$,

$$g'(s) = e^{-s}(1-s).$$

It is positive for $0 \leq s < 1$ and negative for $s > 1$. Thus $g(0) = 0$ increases immediately away from the origin, so

$$(0, 0) \text{ is a strict relative minimum.}$$

The radial profile peaks at $s = 1$, so every point of the unit circle is a relative maximum with value

$$e^{-1}.$$

Step 3: Non-strictness and degeneracy Moving tangentially along the circle leaves s and f unchanged, so none of these maxima is strict. That zero-curvature tangential direction makes the Hessian determinant zero on the critical circle, which is why the standard second derivative test is inconclusive there.

Question 6

Consider the parameterized quadratic

$$f_{a,b}(x, y) = x^2 + axy + by^2 - 4x + 2y.$$

The point $P = (1, -1)$ is known to be critical.

Tasks

1. Determine a and b from the critical-point condition.
2. Classify P for the resulting function.
3. Verify the classification with two lines through P .

Question 6 – Solution

Strategy. Use the two first-derivative equations to reconstruct the parameters, then test the resulting quadratic form around P .

Step 1: Determine the parameters

$$f_x = 2x + ay - 4, \quad f_y = ax + 2by + 2.$$

At $(1, -1)$, the equations $f_x = f_y = 0$ become

$$-2 - a = 0, \quad a - 2b + 2 = 0.$$

Therefore

$$\boxed{a = -2, \quad b = 0}.$$

Step 2: Hessian classification For the resulting function,

$$f_{xx} = 2, \quad f_{yy} = 0, \quad f_{xy} = -2.$$

Thus

$$D = (2)(0) - (-2)^2 = -4 < 0,$$

so P is a saddle point.

Step 3: Direct verification Write $x = 1 + h$, $y = -1 + k$. Since the function is quadratic and P is critical,

$$f(1 + h, -1 + k) - f(1, -1) = h^2 - 2hk.$$

Along $k = 0$, the change is $h^2 > 0$. Along $k = h$, it is $-h^2 < 0$. Hence every neighborhood of P contains values above and below $f(P)$, confirming

$$\boxed{P = (1, -1) \text{ is a saddle point.}}$$

Question 7

A twice differentiable function has the local expansion near a critical point P :

$$f(P + (h, k)) = 5 + 3h^2 + 4hk - k^2 + o(h^2 + k^2).$$

Tasks

1. Reconstruct the Hessian matrix at P .
2. Apply the second derivative test.
3. Use the remainder term carefully to prove the classification along two coordinate directions.

Question 7 – Solution

Strategy. Match the quadratic terms with $\frac{1}{2}(h, k)H(h, k)^T$, then show that the leading signs dominate the smaller remainder.

Step 1: Hessian The second-order Taylor term is

$$\frac{1}{2} (f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2).$$

Matching coefficients gives

$$f_{xx} = 6, \quad f_{xy} = 4, \quad f_{yy} = -2,$$

so

$$H(P) = \begin{pmatrix} 6 & 4 \\ 4 & -2 \end{pmatrix}.$$

Step 2: Second derivative test

$$D = (6)(-2) - 4^2 = -28 < 0.$$

Therefore P is a saddle point.

Step 3: Remainder verification Along $k = 0$,

$$f(P + (h, 0)) - 5 = 3h^2 + o(h^2),$$

which is positive for all sufficiently small nonzero h . Along $h = 0$,

$$f(P + (0, k)) - 5 = -k^2 + o(k^2),$$

which is negative for all sufficiently small nonzero k . The little- o terms are eventually smaller in magnitude than, for example, half of the displayed quadratic terms. Thus nearby values occur on both sides of 5, rigorously confirming the saddle.

Question 8

Let f be differentiable at an interior point $P = (a, b)$ of its domain.

Tasks

1. Prove that if f has a relative maximum or minimum at P , then $\nabla f(P) = \mathbf{0}$.
2. Explain why the converse is false.
3. Apply the result to show that $g(x, y) = x + e^y$ has no relative extrema on $\mathbb{R}^2$.

Question 8 – Solution

Strategy. Restrict the function to coordinate lines through the interior point and apply the one-variable necessary condition.

Step 1: Necessary condition Suppose f has a relative extremum at (a, b) . The one-variable restriction

$$\phi(x) = f(x, b)$$

has a relative extremum at $x = a$. Differentiability gives

$$f_x(a, b) = \phi'(a) = 0.$$

Likewise, $\psi(y) = f(a, y)$ has a relative extremum at b , so

$$f_y(a, b) = \psi'(b) = 0.$$

Therefore

$$\boxed{\nabla f(P) = \mathbf{0}}.$$

Step 2: The converse A zero gradient makes P a critical point, not necessarily an extremum. For instance, $f(x, y) = x^2 - y^2$ has gradient zero at the origin but takes both positive and negative values nearby, so the origin is a saddle.

Step 3: Application For $g(x, y) = x + e^y$,

$$g_x = 1, \quad g_y = e^y.$$

In particular, the gradient is never zero. Since every point of $\mathbb{R}^2$ is interior and g is differentiable everywhere, the necessary condition rules out a relative maximum or minimum at every point. Hence

$$\boxed{g \text{ has no relative extrema on } \mathbb{R}^2}.$$

Question 9

Consider

$$f(x, y) = \sqrt{x^2 + y^2}.$$

Tasks

1. Prove directly that the origin is a strict relative minimum.
2. Determine whether $f_x(0, 0)$ and $f_y(0, 0)$ exist.
3. Explain how this example refines the usual critical-point search procedure.

Question 9 – Solution

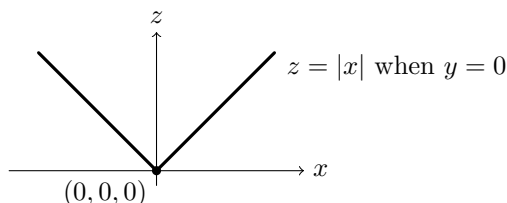
Strategy. Use the definition of a relative minimum first, then test each partial derivative with a signed two-sided limit.

Step 1: Direct extremum test At the origin, $f(0, 0) = 0$. For every $(x, y) \neq (0, 0)$,

$$f(x, y) = \sqrt{x^2 + y^2} > 0 = f(0, 0).$$

Therefore

$(0, 0)$ is a strict relative minimum.



Step 2: Partial derivatives Along the x -axis,

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{|h|}{h},$$

whose right-hand limit is 1 and left-hand limit is -1 . Thus $f_x(0, 0)$ does not exist. The identical calculation on the y -axis shows that $f_y(0, 0)$ does not exist.

Step 3: Methodological conclusion Fermat's theorem says that a *differentiable* interior extremum has zero gradient. It does not say every extremum is differentiable. Consequently, a complete candidate search must include interior points where first partial derivatives vanish *and* points where they fail to exist. This example supplies the latter case.

Question 10

A local profit model, in suitable units, is

$$P(x, y) = 80x + 60y - 2x^2 - xy - y^2.$$

Assume the critical point found below lies in the interior of the operational region.

Tasks

1. Find the critical production levels.
2. Classify the critical point using the Hessian.
3. Complete the square in translated coordinates to verify the local classification and compute the modeled profit there.

Question 10 – Solution

Strategy. Solve the marginal-profit equations, then verify the Hessian conclusion by expressing every nearby change as a negative quadratic form.

Step 1: Critical production levels

$$P_x = 80 - 4x - y, \quad P_y = 60 - x - 2y.$$

The equations $4x + y = 80$ and $x + 2y = 60$ yield

$$\boxed{x = \frac{100}{7}, \quad y = \frac{160}{7}}.$$

Step 2: Hessian test

$$P_{xx} = -4, \quad P_{yy} = -2, \quad P_{xy} = -1.$$

Thus

$$D = (-4)(-2) - (-1)^2 = 7 > 0.$$

Because $P_{xx} < 0$, the critical point is a strict relative maximum. Direct substitution gives

$$\boxed{P\left(\frac{100}{7}, \frac{160}{7}\right) = \frac{8800}{7}}.$$

Step 3: Translated verification Let $h = x - 100/7$ and $k = y - 160/7$. Since P is quadratic and its linear terms vanish at the critical point,

$$P(x, y) - \frac{8800}{7} = -2h^2 - hk - k^2.$$

Complete the square:

$$-2h^2 - hk - k^2 = -2\left(h + \frac{k}{4}\right)^2 - \frac{7}{8}k^2 < 0$$

for every $(h, k) \neq (0, 0)$. This independently verifies the strict relative maximum.