

**Question 1**

Consider the ellipsoid

$$F(x, y, z) = x^2 + 2y^2 + 3z^2 = 6$$

at  $P = (1, 1, 1)$ .

**Tasks**

1. Compute the gradient and verify that  $P$  is a regular point.
2. Find the tangent plane at  $P$ .
3. Find a parametric equation of the normal line and verify its direction.

## Question 1 – Solution

**Strategy.** For a regular level surface, the gradient at the point is normal to the tangent plane.

**Step 1: Gradient and regularity**

$$\nabla F = \langle 2x, 4y, 6z \rangle, \quad \nabla F(1, 1, 1) = \langle 2, 4, 6 \rangle.$$

This vector is nonzero, so  $P$  is regular. Also  $F(1, 1, 1) = 1 + 2 + 3 = 6$ , confirming that  $P$  lies on the surface.

**Step 2: Tangent plane** Using  $\nabla F(P)$  as a normal,

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

After division by 2,

$$\boxed{x + 2y + 3z = 6}.$$

**Step 3: Normal line** A simplified direction parallel to the gradient is  $\langle 1, 2, 3 \rangle$ . Hence

$$\boxed{(x, y, z) = (1, 1, 1) + t(1, 2, 3)},$$

or  $x = 1 + t$ ,  $y = 1 + 2t$ ,  $z = 1 + 3t$ .

**Verification** The line passes through  $P$  at  $t = 0$ , and  $\langle 1, 2, 3 \rangle$  is perpendicular to every displacement  $\langle u, v, w \rangle$  in the plane because the plane equation requires  $u + 2v + 3w = 0$ .

**Question 2**

Let the level surface

$$F(x, y, z) = xye^z = 2$$

contain  $P = (1, 2, 0)$ .

**Tasks**

1. Find the two unit normal vectors at  $P$ .
2. Write the tangent plane in standard form.
3. Give symmetric or parametric equations for the normal line.

## Question 2 – Solution

**Strategy.** Differentiate the defining level function; its gradient supplies both the plane normal and the normal-line direction.

**Step 1: Gradient**

$$\nabla F = \langle ye^z, xe^z, xye^z \rangle.$$

At  $P = (1, 2, 0)$ ,

$$\nabla F(P) = \langle 2, 1, 2 \rangle, \quad \|\nabla F(P)\| = 3.$$

Therefore the two unit normals are

$$\pm \frac{1}{3} \langle 2, 1, 2 \rangle.$$

**Step 2: Tangent plane**

$$2(x - 1) + (y - 2) + 2(z - 0) = 0,$$

so

$$2x + y + 2z = 4.$$

**Step 3: Normal line** Using the unsimplified integer direction,

$$(x, y, z) = (1, 2, 0) + t(2, 1, 2).$$

Equivalently,

$$\frac{x - 1}{2} = y - 2 = \frac{z}{2}.$$

**Verification** The point satisfies  $1 \cdot 2 \cdot e^0 = 2$ . The line direction equals the plane's coefficient vector, so it is perpendicular to the plane.

**Question 3**

The sphere

$$x^2 + y^2 + z^2 = 9$$

and the plane

$$x + y + z = 5$$

intersect in a curve through  $P = (1, 2, 2)$ .

**Tasks**

1. Find a direction tangent to the intersection curve at  $P$ .
2. Write the tangent line to the curve.
3. Verify that the direction is tangent to both surfaces.

### Question 3 – Solution

**Strategy.** A curve lying on both surfaces has a tangent perpendicular to both surface normals, so use their cross product.

**Step 1: Surface normals** Let

$$F = x^2 + y^2 + z^2, \quad G = x + y + z.$$

At  $P$ ,

$$\nabla F(P) = \langle 2, 4, 4 \rangle, \quad \nabla G(P) = \langle 1, 1, 1 \rangle.$$

Their cross product is

$$\nabla F(P) \times \nabla G(P) = \langle 0, 2, -2 \rangle,$$

so a simplified tangent direction is  $\langle 0, 1, -1 \rangle$ .

**Step 2: Tangent line**

$$\boxed{(x, y, z) = (1, 2, 2) + t(0, 1, -1)}.$$

**Step 3: Verification** First,  $1^2 + 2^2 + 2^2 = 9$  and  $1 + 2 + 2 = 5$ , so  $P$  lies on both surfaces. Next,

$$\langle 2, 4, 4 \rangle \cdot \langle 0, 1, -1 \rangle = 0, \quad \langle 1, 1, 1 \rangle \cdot \langle 0, 1, -1 \rangle = 0.$$

Thus the direction lies in both tangent planes, as required.

**Question 4**

On the ellipsoid

$$x^2 + 2y^2 + z^2 = 9,$$

find every point whose tangent plane is parallel to

$$2x - 4y + 2z = 7.$$

**Tasks**

1. Determine all points of tangency.
2. Write the tangent plane at each point.
3. Give a normal line at each point and verify completeness.

## Question 4 – Solution

**Strategy.** Parallel tangent planes have parallel normal vectors; impose that condition together with the surface equation.

**Step 1: Parallel-gradient condition** For  $F = x^2 + 2y^2 + z^2$ ,

$$\nabla F = \langle 2x, 4y, 2z \rangle.$$

The given plane has normal parallel to  $\langle 1, -2, 1 \rangle$ . Set

$$\langle 2x, 4y, 2z \rangle = \lambda \langle 1, -2, 1 \rangle.$$

Then  $x = \lambda/2$ ,  $y = -\lambda/2$ , and  $z = \lambda/2$ . Substitution into the ellipsoid gives  $\lambda^2 = 9$ .

Thus the two points are

$$P_+ = \left( \frac{3}{2}, -\frac{3}{2}, \frac{3}{2} \right), \quad P_- = -P_+.$$

**Step 2: Tangent planes** Using the simplified common normal,

$$x - 2y + z = 6 \quad \text{at } P_+,$$

$$x - 2y + z = -6 \quad \text{at } P_-.$$

**Step 3: Normal lines**

$$\mathbf{r} = P_+ + t \langle 1, -2, 1 \rangle, \quad \mathbf{r} = P_- + t \langle 1, -2, 1 \rangle.$$

The equations  $\lambda = \pm 3$  exhaust the surface constraint, proving there are no additional points.

**Question 5**

For the surface

$$xy + yz + zx = 3,$$

consider  $P = (1, 1, 1)$ .

**Tasks**

1. Find the normal line at  $P$ .
2. Find every point where that normal line intersects the sphere  $x^2 + y^2 + z^2 = 12$ .
3. Verify both the surface-normal direction and the sphere intersections.

## Question 5 – Solution

**Strategy.** Parameterize the normal line using the gradient, then substitute that line into the sphere equation.

**Step 1: Normal line** Let  $F = xy + yz + zx$ . Then

$$\nabla F = \langle y + z, x + z, x + y \rangle, \quad \nabla F(P) = \langle 2, 2, 2 \rangle.$$

Using the simplified direction  $\langle 1, 1, 1 \rangle$ ,

$$\boxed{(x, y, z) = (1 + t, 1 + t, 1 + t)}.$$

**Step 2: Intersections with the sphere** Substitution gives

$$3(1 + t)^2 = 12,$$

so  $(1 + t)^2 = 4$  and

$$t = 1 \quad \text{or} \quad t = -3.$$

The intersection points are therefore

$$\boxed{(2, 2, 2) \quad \text{and} \quad (-2, -2, -2)}.$$

**Step 3: Verification** The base point satisfies  $1 + 1 + 1 = 3$ , and the line direction is parallel to  $\nabla F(P)$ , so the line is normal to the original surface. Finally,

$$2^2 + 2^2 + 2^2 = 12, \quad (-2)^2 + (-2)^2 + (-2)^2 = 12,$$

confirming both sphere intersections.

**Question 6**

Consider the double cone

$$F(x, y, z) = x^2 + y^2 - z^2 = 0.$$

**Tasks**

1. Find the tangent plane and normal line at the regular point  $P = (1, 0, 1)$ .
2. Explain why the gradient formula fails to select a plane at the origin.
3. Use generator curves through the origin to prove that no single tangent plane contains all tangent directions there.

## Question 6 – Solution

**Strategy.** Contrast a regular point, where the gradient is nonzero, with the cone vertex, where many generator directions meet and the gradient vanishes.

**Step 1: The regular point**

$$\nabla F = \langle 2x, 2y, -2z \rangle, \quad \nabla F(1, 0, 1) = \langle 2, 0, -2 \rangle.$$

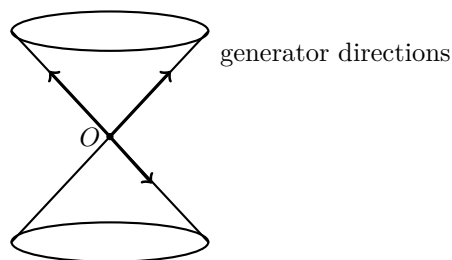
Thus

$$2(x - 1) - 2(z - 1) = 0, \quad \boxed{x - z = 0},$$

and a normal line is

$$\boxed{(x, y, z) = (1, 0, 1) + t(1, 0, -1)}.$$

**Step 2: The vertex** At  $O = (0, 0, 0)$ ,  $\nabla F(O) = \langle 0, 0, 0 \rangle$ , so the usual plane equation reduces to  $0 = 0$  and supplies no normal.



**Step 3: Generator argument** For every  $\theta$ , the curve

$$\mathbf{r}_\theta(s) = \langle s \cos \theta, s \sin \theta, s \rangle$$

lies on the cone and has tangent  $\langle \cos \theta, \sin \theta, 1 \rangle$  at  $O$ . The directions for  $\theta = 0, \pi$ , and  $\pi/2$  span all of  $\mathbb{R}^3$ : the first two yield the  $x$ - and  $z$ -directions by differences and sums, and the third then yields the  $y$ -direction. No two-dimensional plane can contain them all. Hence  $\boxed{\text{there is no unique tangent plane at the vertex}}.$

**Question 7**

An unknown member of the family

$$F(x, y, z) = x^2 + ay^2 + bz^2 = c$$

passes through  $P = (1, 1, 1)$  and has tangent plane

$$x + 2y + 3z = 6$$

there.

**Tasks**

1. Determine  $a$ ,  $b$ , and  $c$ .
2. Confirm that  $P$  is regular for the resulting surface.
3. Write its normal line at  $P$  and verify the prescribed tangent plane.

## Question 7 – Solution

**Strategy.** Match the unknown gradient to the plane's normal; the first gradient component fixes the proportionality factor.

**Step 1: Match normals**

$$\nabla F(P) = \langle 2, 2a, 2b \rangle.$$

This must equal  $\lambda \langle 1, 2, 3 \rangle$ . The first component gives  $\lambda = 2$ . Therefore

$$2a = 4, \quad 2b = 6,$$

so

$$\boxed{a = 2, \quad b = 3}.$$

Since  $P$  lies on the surface,

$$c = 1^2 + 2(1)^2 + 3(1)^2 = \boxed{6}.$$

**Step 2: Regularity** For the resulting surface,

$$\nabla F(P) = \langle 2, 4, 6 \rangle \neq \langle 0, 0, 0 \rangle,$$

so  $P$  is regular.

**Step 3: Line and plane verification** The normal line is

$$\boxed{(x, y, z) = (1, 1, 1) + t(1, 2, 3)}.$$

The plane obtained from the gradient is

$$2(x - 1) + 4(y - 1) + 6(z - 1) = 0.$$

Dividing by 2 produces  $x + 2y + 3z = 6$ , exactly the prescribed plane.

**Question 8**

For the level surface

$$xyz = 8,$$

let  $P = (2, 2, 2)$ .

**Tasks**

1. Find the tangent plane and normal line at  $P$ .
2. Show that the normal line passes through the origin.
3. Prove that  $P$  is the perpendicular foot from the origin to the tangent plane and find that distance in two ways.

## Question 8 – Solution

**Strategy.** Use the gradient for the plane and line, then compare the point-to-plane formula with the length of the normal segment.

**Step 1: Plane and line** For  $F = xyz$ ,

$$\nabla F = \langle yz, xz, xy \rangle, \quad \nabla F(P) = \langle 4, 4, 4 \rangle.$$

Thus the tangent plane is

$$4(x - 2) + 4(y - 2) + 4(z - 2) = 0,$$

or

$$\boxed{x + y + z = 6}.$$

A normal line is

$$\boxed{(x, y, z) = (2, 2, 2) + t(1, 1, 1)}.$$

**Step 2: Origin on the line** At  $t = -2$ , all three coordinates are zero. Thus the segment from the origin to  $P$  follows the plane's normal direction, making  $P$  the perpendicular foot.

**Step 3: Distance checks** The normal-segment length is

$$\| \langle 2, 2, 2 \rangle \| = \boxed{2\sqrt{3}}.$$

The point-to-plane formula independently gives

$$\frac{|0 + 0 + 0 - 6|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{6}{\sqrt{3}} = \boxed{2\sqrt{3}}.$$

The agreement verifies both the geometry and the scale of the normal vector.

**Question 9**

Let  $S$  be a smooth level surface  $F(x, y, z) = c$ , and let  $P$  be a point of  $S$  with  $\nabla F(P) \neq 0$ .

**Tasks**

1. For any differentiable curve  $\mathbf{r}(t)$  on  $S$  through  $P$ , prove that its tangent at  $P$  is perpendicular to  $\nabla F(P)$ .
2. Deduce the tangent-plane equation at  $P = (x_0, y_0, z_0)$ .
3. Deduce the normal-line equation and explain the role of the regularity condition.

## Question 9 – Solution

**Strategy.** Differentiate the constant identity  $F(\mathbf{r}(t)) = c$  and interpret the chain rule geometrically.

**Step 1: Tangent vectors** Suppose  $\mathbf{r}(t_0) = P$  and the curve lies on  $S$ . Then

$$F(\mathbf{r}(t)) = c.$$

Differentiating at  $t_0$  by the multivariable chain rule gives

$$0 = \left. \frac{d}{dt} F(\mathbf{r}(t)) \right|_{t=t_0} = \nabla F(P) \cdot \mathbf{r}'(t_0).$$

Thus every surface-tangent velocity is perpendicular to  $\nabla F(P)$ .

**Step 2: Tangent plane** Because  $\nabla F(P)$  is nonzero, it supplies a valid normal. Therefore

$$\boxed{F_x(P)(x - x_0) + F_y(P)(y - y_0) + F_z(P)(z - z_0) = 0}.$$

**Step 3: Normal line** The line through  $P$  in the gradient direction is

$$\boxed{\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \nabla F(P)}.$$

The condition  $\nabla F(P) \neq 0$  ensures a genuine normal direction and a two-dimensional tangent plane. At a singular point, the zero vector cannot determine either object; additional geometric analysis is required.

**Question 10**

On the ellipsoid

$$x^2 + 2y^2 + 3z^2 = 6,$$

find every point at which the normal line passes through the origin.

**Tasks**

1. Derive the condition relating a point to its gradient.
2. Prove that at most one coordinate of such a point can be nonzero.
3. List all points, tangent planes, and normal lines, and verify completeness.

## Question 10 – Solution

**Strategy.** If the origin lies on the normal line at  $P$ , then the vector from  $P$  to the origin must be parallel to  $\nabla F(P)$ .

**Step 1: Parallelism condition** Write  $P = (p, q, r)$ . Since

$$\nabla F(P) = \langle 2p, 4q, 6r \rangle,$$

the origin lies on the normal line exactly when some  $t$  satisfies

$$\langle -p, -q, -r \rangle = t \langle 2p, 4q, 6r \rangle.$$

If  $p \neq 0$ , then  $t = -1/2$ ; if  $q \neq 0$ , then  $t = -1/4$ ; if  $r \neq 0$ , then  $t = -1/6$ . One value of  $t$  cannot meet two of these conditions, so at most one coordinate is nonzero.

**Step 2: Points and planes** Using the ellipsoid equation on each coordinate axis gives exactly six points:

$$(\pm\sqrt{6}, 0, 0), \quad (0, \pm\sqrt{3}, 0), \quad (0, 0, \pm\sqrt{2}).$$

Their tangent planes are, respectively,

$$\boxed{x = \pm\sqrt{6}}, \quad \boxed{y = \pm\sqrt{3}}, \quad \boxed{z = \pm\sqrt{2}}.$$

**Step 3: Normal lines and verification** The corresponding normal lines are

$$\boxed{\mathbf{r} = (\pm\sqrt{6}, 0, 0) + t \langle 1, 0, 0 \rangle},$$
$$\boxed{\mathbf{r} = (0, \pm\sqrt{3}, 0) + t \langle 0, 1, 0 \rangle}, \quad \boxed{\mathbf{r} = (0, 0, \pm\sqrt{2}) + t \langle 0, 0, 1 \rangle}.$$

These are the coordinate axes written through their stated points. Every one passes through the origin and is perpendicular to its listed coordinate plane. The parallelism argument rules out mixed-coordinate points, while the surface equation rules out the origin itself, so the list is complete.