

Question 1

For the graph

$$z = f(x, y) = x^2 e^y,$$

consider the point above $(1, 0)$.

Tasks

1. Find the tangent plane and the linearization $L(x, y)$ at $(1, 0)$.
2. Use L to estimate $f(1.02, -0.03)$.
3. Compare the estimate with the calculator value $1.0404e^{-0.03}$.

Question 1 – Solution

Strategy. Use the value and both first partial derivatives at the base point to build the unique affine approximation.

Step 1: Local data

$$f(1, 0) = 1, \quad f_x = 2xe^y, \quad f_y = x^2e^y.$$

Thus $f_x(1, 0) = 2$ and $f_y(1, 0) = 1$.

Step 2: Plane and linearization The tangent-plane formula gives

$$z - 1 = 2(x - 1) + (y - 0).$$

Consequently,

$$\boxed{z = 1 + 2(x - 1) + y}, \quad \boxed{L(x, y) = 1 + 2(x - 1) + y}.$$

Step 3: Estimate Here $\Delta x = 0.02$ and $\Delta y = -0.03$, so

$$f(1.02, -0.03) \approx L(1.02, -0.03) = 1 + 2(0.02) - 0.03 = \boxed{1.0100}.$$

Verification Direct evaluation gives

$$1.0404e^{-0.03} \approx 1.0096515.$$

The linearization is high by about 0.0003485, a small error consistent with the small displacement from $(1, 0)$.

Question 2

The upper hemisphere

$$z = \sqrt{25 - x^2 - y^2}$$

is sampled near the point $(3, 0, 4)$.

Tasks

1. Derive the linearization at $(x, y) = (3, 0)$.
2. Estimate the height above $(3.04, 0.06)$.
3. Explain geometrically why the first-order estimate has no y -correction.

Question 2 – Solution

Strategy. Differentiate the height function, then evaluate its first-order change from the base point.

Step 1: Partial derivatives For $f(x, y) = (25 - x^2 - y^2)^{1/2}$,

$$f_x = -\frac{x}{\sqrt{25 - x^2 - y^2}}, \quad f_y = -\frac{y}{\sqrt{25 - x^2 - y^2}}.$$

At $(3, 0)$, these become $f_x = -3/4$ and $f_y = 0$.

Step 2: Linearization

$$L(x, y) = 4 - \frac{3}{4}(x - 3).$$

At the requested point,

$$f(3.04, 0.06) \approx 4 - \frac{3}{4}(0.04) = \boxed{3.9700}.$$

Step 3: Geometry The sphere is symmetric under $y \mapsto -y$. At $y = 0$, moving a small distance in either y -direction changes the height only to second order, so the tangent plane has no y -tilt.

Verification The exact height is

$$\sqrt{25 - 3.04^2 - 0.06^2} = \sqrt{15.7548} \approx 3.9692317.$$

The estimate is high by about 0.0007683.

Question 3

A differentiable function f has tangent plane

$$z = 7 + 3(x - 2) - 4(y + 1)$$

at the point whose input coordinates are $(2, -1)$.

Tasks

1. Recover $f(2, -1)$, $f_x(2, -1)$, and $f_y(2, -1)$.
2. Predict the first-order change along $x = 2 + t$, $y = -1 + 2t$.
3. Construct one nonlinear polynomial f having exactly this tangent plane.

Question 3 – Solution

Strategy. Read the constant and slope coefficients from the standard linearization form, then add terms that vanish to first order.

Step 1: Recover the data Comparing with

$$L = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

at $(a, b) = (2, -1)$ gives

$$\boxed{f(2, -1) = 7}, \quad \boxed{f_x(2, -1) = 3}, \quad \boxed{f_y(2, -1) = -4}.$$

Step 2: Change along the path The input changes are $\Delta x = t$ and $\Delta y = 2t$. Hence

$$\Delta f \approx 3t - 4(2t) = \boxed{-5t}.$$

Step 3: Construct an example One choice is

$$\boxed{f(x, y) = 7 + 3(x - 2) - 4(y + 1) + (x - 2)^2 + (y + 1)^2}.$$

The two quadratic terms and their first partial derivatives vanish at $(2, -1)$, so they do not alter the tangent plane.

Verification Substitution gives $f(2, -1) = 7$. Differentiation gives $f_x = 3 + 2(x - 2)$ and $f_y = -4 + 2(y + 1)$, which reproduce the recovered slopes at the base point.

Question 4

For the graph

$$z = f(x, y) = x^3 - 3x + y^2 - 4y,$$

locate every point at which the tangent plane is horizontal.

Tasks

1. Translate “horizontal tangent plane” into conditions on the partial derivatives.
2. Find all corresponding points on the graph.
3. Write each horizontal tangent plane and show that no others exist.

Question 4 – Solution

Strategy. A tangent plane to a graph is horizontal precisely when both of its slope coefficients vanish.

Step 1: Slope conditions

$$f_x = 3x^2 - 3, \quad f_y = 2y - 4.$$

Thus a horizontal plane requires

$$3x^2 - 3 = 0, \quad 2y - 4 = 0.$$

Therefore $x = \pm 1$ and $y = 2$.

Step 2: Heights

$$f(1, 2) = 1 - 3 + 4 - 8 = -6,$$

$$f(-1, 2) = -1 + 3 + 4 - 8 = -2.$$

The two points are

$$\boxed{(1, 2, -6) \quad \text{and} \quad (-1, 2, -2)}.$$

Step 3: Planes and completeness Because both slope coefficients are zero, the planes are

$$\boxed{z = -6} \quad \text{and} \quad \boxed{z = -2}.$$

The equations $x^2 = 1$ and $y = 2$ exhaust every simultaneous zero of f_x and f_y , so there are no other horizontal tangent planes.

Question 5

Define

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute both partial derivatives at the origin from their definitions.
2. Identify the candidate tangent plane suggested by those partial derivatives.
3. Test the linear-approximation remainder along $y = x^2$ and decide whether a tangent plane exists there in the differentiability sense.

Question 5 – Solution

Strategy. Existing partial derivatives only suggest a plane; differentiability requires the remainder to be negligible compared with the distance to the base point.

Step 1: Axis partial derivatives Along either coordinate axis, f is zero. Hence

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - 0}{h} = 0, \quad f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k) - 0}{k} = 0.$$

Step 2: Candidate plane The value and both proposed slopes are zero, so the formal candidate is

$$\boxed{z = 0}.$$

Step 3: Remainder test Along $(x, y) = (t, t^2)$ with $t \neq 0$,

$$f(t, t^2) = \frac{t^4}{t^4 + t^4} = \frac{1}{2}.$$

For the candidate linearization $L = 0$, the required quotient is

$$\frac{|f(t, t^2) - L(t, t^2)|}{\sqrt{t^2 + t^4}} = \frac{1/2}{|t|\sqrt{1 + t^2}} \rightarrow \infty.$$

Therefore the error is not $o(\sqrt{x^2 + y^2})$.

Conclusion The function is not even continuous at the origin along this curve, so it is not differentiable there. Thus the graph has $\boxed{\text{no tangent plane at the origin in the linear-approximation sense}}.$

Question 6

Let

$$f(x, y) = e^{x+2y}.$$

Use the linearization at $(0, 0)$ to estimate $f(0.02, 0.01)$ and rigorously bound the absolute error using the one-variable Taylor theorem for e^t .

Tasks

1. Find the linearization and the estimate.
2. Obtain a uniform error bound for $|x| \leq 0.02$, $|y| \leq 0.02$.
3. Compare the actual error at $(0.02, 0.01)$ with that bound.

Question 6 – Solution

Strategy. Write $t = x + 2y$ so the two-variable approximation reduces to the first-order Taylor approximation of e^t at zero.

Step 1: Linearization At the origin, $f = 1$, $f_x = 1$, and $f_y = 2$. Therefore

$$\boxed{L(x, y) = 1 + x + 2y}.$$

At $(0.02, 0.01)$, $t = 0.04$, so

$$f(0.02, 0.01) \approx 1.04.$$

Step 2: Uniform bound Taylor's theorem gives

$$e^t = 1 + t + \frac{e^\xi}{2}t^2$$

for some ξ between 0 and t . In the given rectangle,

$$|t| = |x + 2y| \leq 0.06,$$

so $e^\xi \leq e^{0.06}$. Consequently,

$$\boxed{|f - L| \leq \frac{e^{0.06}}{2}(0.06)^2 < 0.001912}.$$

Step 3: Actual error

$$|e^{0.04} - 1.04| \approx \boxed{0.0008108},$$

which lies below the uniform bound. The bound is larger because it must cover the entire rectangle, not only the requested point.

Question 7

For the paraboloid

$$z = x^2 + 2y^2,$$

find the tangent plane parallel to the plane

$$2x - 4y + z = 7.$$

Tasks

1. Determine the point of tangency.
2. Write the tangent plane in standard form.
3. Find the perpendicular distance between the two parallel planes.

Question 7 – Solution

Strategy. Match the x - and y -slope coefficients of the tangent plane to those of the given plane.

Step 1: Match slopes The given plane is $z = 7 - 2x + 4y$, so its slopes are -2 and 4 . For $f = x^2 + 2y^2$,

$$f_x = 2x, \quad f_y = 4y.$$

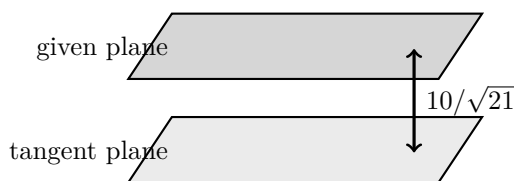
Thus $2x = -2$ and $4y = 4$, giving $(x, y) = (-1, 1)$. The height is $z = 1 + 2 = 3$.

Step 2: Tangent plane At $(-1, 1, 3)$,

$$z - 3 = -2(x + 1) + 4(y - 1),$$

or

$$\boxed{2x - 4y + z = -3}.$$



Step 3: Separation For parallel planes $2x - 4y + z = 7$ and $2x - 4y + z = -3$, the distance is

$$\boxed{\frac{|7 - (-3)|}{\sqrt{2^2 + (-4)^2 + 1^2}} = \frac{10}{\sqrt{21}}}.$$

The identical left sides verify parallelism, and substituting $(-1, 1, 3)$ into the tangent plane gives -3 .

Question 8

Suppose a function f is differentiable at (a, b) . An affine function has the form

$$A(x, y) = c + p(x - a) + q(y - b).$$

Tasks

1. Prove that if A is a linear approximation to f at (a, b) , then $c = f(a, b)$.
2. Use displacements along the coordinate axes to prove $p = f_x(a, b)$ and $q = f_y(a, b)$.
3. Conclude that the tangent-plane linearization is unique.

Question 8 – Solution

Strategy. Apply the defining small-error condition first at zero displacement and then on each coordinate axis.

Step 1: Constant term For A to approximate f at the base point, it must agree there. Since $A(a, b) = c$,

$$\boxed{c = f(a, b)}.$$

Equivalently, the remainder must vanish when the displacement is zero.

Step 2: The x coefficient The approximation condition along $(h, 0)$ is

$$f(a + h, b) = f(a, b) + ph + o(|h|).$$

Subtract $f(a, b)$, divide by $h \neq 0$, and let $h \rightarrow 0$. Since $o(|h|)/h \rightarrow 0$ from both sides,

$$p = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h} = \boxed{f_x(a, b)}.$$

Step 3: The y coefficient The same argument along $(0, k)$ yields

$$q = \lim_{k \rightarrow 0} \frac{f(a, b + k) - f(a, b)}{k} = \boxed{f_y(a, b)}.$$

Conclusion Every affine first-order approximation must therefore be

$$\boxed{L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}.$$

All three coefficients are forced, proving uniqueness.

Question 9

A cylinder has volume

$$V(r, h) = \pi r^2 h.$$

Its nominal dimensions are $r = 10$ cm and $h = 20$ cm. A measured cylinder instead has $\Delta r = 0.03$ cm and $\Delta h = -0.08$ cm.

Tasks

1. Use linearization to estimate the change in volume.
2. Estimate the relative and percentage changes.
3. Compute the exact change and quantify the linearization error.

Question 9 – Solution

Strategy. Evaluate the tangent-plane increment $\Delta V \approx V_r \Delta r + V_h \Delta h$ at the nominal dimensions.

Step 1: Volume change

$$V_r = 2\pi r h, \quad V_h = \pi r^2.$$

At (10, 20),

$$dV = (400\pi)(0.03) + (100\pi)(-0.08) = 12\pi - 8\pi.$$

Thus

$$\boxed{\Delta V \approx 4\pi \text{ cm}^3 \approx 12.566 \text{ cm}^3}.$$

Step 2: Relative change Dividing the differential by $V = \pi r^2 h$ gives

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2\frac{0.03}{10} - \frac{0.08}{20} = 0.002.$$

The estimated percentage increase is $\boxed{0.2\%}$.

Step 3: Exact comparison

$$\Delta V_{\text{exact}} = \pi(10.03)^2(19.92) - 2000\pi = \boxed{3.969928\pi \text{ cm}^3}.$$

The linear estimate exceeds the exact change by

$$(4 - 3.969928)\pi = \boxed{0.030072\pi \text{ cm}^3 \approx 0.09447 \text{ cm}^3}.$$

Question 10

Consider the family

$$f_{a,b}(x, y) = ax^2 + bxy + y^2.$$

At $(1, 1)$, its tangent plane passes through $(2, 0, 5)$, and its linearization predicts the value 3.06 at $(1.02, 0.99)$.

Tasks

1. Determine the parameters a and b .
2. Write the resulting tangent plane at $(1, 1)$.
3. Use it to estimate $f(0.98, 1.03)$ and verify both original constraints.

Question 10 – Solution

Strategy. Express the value and slopes at $(1, 1)$ in terms of a, b , then turn the two tangent-plane conditions into a linear system.

Step 1: General linearization At $(1, 1)$,

$$f = a + b + 1, \quad f_x = 2a + b, \quad f_y = b + 2.$$

Therefore

$$L(x, y) = a + b + 1 + (2a + b)(x - 1) + (b + 2)(y - 1).$$

Passing through $(2, 0, 5)$ requires

$$3a + b - 1 = 5, \quad \text{so} \quad 3a + b = 6.$$

The prediction at $(1.02, 0.99)$ requires

$$1.04a + 1.01b + 0.98 = 3.06.$$

Solving the two equations gives

$$\boxed{a = 2, \quad b = 0}.$$

Step 2: Tangent plane Now $f(1, 1) = 3$, $f_x(1, 1) = 4$, and $f_y(1, 1) = 2$. Hence

$$\boxed{z = 3 + 4(x - 1) + 2(y - 1)}.$$

Step 3: New estimate and verification At $(0.98, 1.03)$,

$$f(0.98, 1.03) \approx 3 + 4(-0.02) + 2(0.03) = \boxed{2.98}.$$

Also $L(2, 0) = 3 + 4 - 2 = 5$, and $L(1.02, 0.99) = 3 + 0.08 - 0.02 = 3.06$, verifying both supplied constraints.