

Question 1

For

$$f(x, y) = x^2y + e^{x-y},$$

find the directional derivative at $P = (1, 0)$ in the direction $v = \langle 3, 4 \rangle$.

Tasks

1. Normalize the direction.
2. Compute the gradient at P .
3. Evaluate and interpret the directional derivative.

Question 1 – Solution

Strategy. A directional derivative uses a unit vector: $D_u f = \nabla f \cdot u$.

Step 1: Normalize

$$\|v\| = 5, \quad u = \boxed{\langle 3/5, 4/5 \rangle}.$$

Step 2: Gradient

$$f_x = 2xy + e^{x-y}, \quad f_y = x^2 - e^{x-y}.$$

At $(1, 0)$,

$$\nabla f(1, 0) = \langle e, 1 - e \rangle.$$

Step 3: Dot product

$$D_u f(1, 0) = e \frac{3}{5} + (1 - e) \frac{4}{5} = \boxed{\frac{4 - e}{5}}.$$

The positive value means f initially increases by approximately $(4 - e)/5$ output units per unit distance traveled in the specified direction.

Question 2

Let $g(x, y) = \ln(x^2 + y^2)$. Find its rate of change at $P = (1, -2)$ toward $Q = (4, 2)$.

Tasks

1. Construct the unit vector from P to Q .
2. Compute the directional derivative exactly.
3. Decide whether g initially increases or decreases.

Question 2 – Solution

Strategy. Subtract endpoint coordinates in the correct order before normalizing.

Step 1: Direction

$$\overrightarrow{PQ} = \langle 3, 4 \rangle, \quad u = \langle 3/5, 4/5 \rangle.$$

Step 2: Gradient

$$\nabla g = \left\langle \frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2} \right\rangle,$$

so

$$\nabla g(1, -2) = \langle 2/5, -4/5 \rangle.$$

Therefore

$$D_u g(P) = \frac{2}{5} \frac{3}{5} - \frac{4}{5} \frac{4}{5} = \boxed{-\frac{2}{5}}.$$

Step 3: Interpret The negative sign means g decreases initially toward Q . This is consistent with the initial motion having a component toward smaller distance from the origin.

Question 3

At a point P , a differentiable function has $\nabla f(P) = \langle -2, 5 \rangle$.

Tasks

1. Find the maximum and minimum directional derivatives and their directions.
2. Find all unit directions giving zero directional derivative.
3. Find the directional derivative at angle $\pi/6$ from the positive x -axis.

Question 3 – Solution

Strategy. Use $D_u f = \|\nabla f\| \cos \alpha$, where α is the angle from the gradient to the unit direction.

Step 1: Extremes Since $\|\nabla f\| = \sqrt{29}$,

$$D_{\max} = \sqrt{29} \text{ in direction } \frac{1}{\sqrt{29}} \langle -2, 5 \rangle,$$

$$D_{\min} = -\sqrt{29} \text{ in direction } \frac{1}{\sqrt{29}} \langle 2, -5 \rangle.$$

Step 2: Zero directions Unit vectors perpendicular to $\langle -2, 5 \rangle$ are

$$u = \pm \frac{1}{\sqrt{29}} \langle 5, 2 \rangle.$$

Step 3: Prescribed angle For $u = \langle \sqrt{3}/2, 1/2 \rangle$,

$$D_u f = -2 \frac{\sqrt{3}}{2} + 5 \frac{1}{2} = \frac{5}{2} - \sqrt{3}.$$

Question 4

At P , $\nabla f(P) = \langle 6, 8 \rangle$. Determine every unit direction u for which $D_u f(P) = 5$.

Tasks

1. Convert the condition to geometry.
2. Find both unit vectors exactly.
3. Verify normalization and the required rate.

Question 4 – Solution

Strategy. Decompose the desired unit vector into components parallel and perpendicular to the gradient.

Step 1: Unit basis Let

$$e = \frac{1}{10} \langle 6, 8 \rangle = \langle 3/5, 4/5 \rangle, \quad n = \langle -4/5, 3/5 \rangle.$$

These form an orthonormal basis. Write $u = ae + bn$ with $a^2 + b^2 = 1$.

Step 2: Rate condition

$$\nabla f \cdot u = 10a = 5,$$

so $a = 1/2$ and $b = \pm\sqrt{3}/2$. Thus

$$u = \frac{1}{2} \langle 3/5, 4/5 \rangle \pm \frac{\sqrt{3}}{2} \langle -4/5, 3/5 \rangle.$$

Equivalently,

$$u = \left\langle \frac{3 \mp 4\sqrt{3}}{10}, \frac{4 \pm 3\sqrt{3}}{10} \right\rangle.$$

Step 3: Verify Orthonormality gives $\|u\|^2 = 1/4 + 3/4 = 1$, and the perpendicular component contributes zero to the dot product, leaving $10(1/2) = 5$.

Question 5

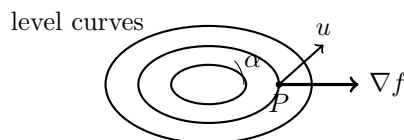
Let f be continuously differentiable near P . At P , its level curve is smooth and $\nabla f(P) \neq 0$. A unit direction u makes angle α with $\nabla f(P)$.

Tasks

1. Derive $D_u f(P)$ in terms of α .
2. Explain why tangent motion along the level curve gives zero rate.
3. Determine the sign of the rate for acute, right, and obtuse α .

Question 5 – Solution

Strategy. Combine the dot-product formula with the gradient's normal relationship to level curves.



Step 1: Angular formula

$$D_u f(P) = \nabla f(P) \cdot u = \|\nabla f(P)\| \cos \alpha.$$

Step 2: Tangent direction Along the level curve, f is constant, so its rate is zero. Geometrically the tangent is perpendicular to ∇f , giving $\alpha = \pi/2$ and $\cos \alpha = 0$.

Step 3: Signs The rate is positive for acute α , zero for a right angle, and negative for obtuse α . Its magnitude is greatest when u is parallel or antiparallel to the gradient.

Question 6

For the scalar field

$$F(x, y, z) = xe^{yz} + z^2,$$

find the directional derivative at $P = (2, 0, 1)$ toward $Q = (-1, 2, 3)$.

Tasks

1. Find the three-dimensional unit direction.
2. Compute $\nabla F(P)$.
3. Evaluate the rate and verify its units conceptually.

Question 6 – Solution

Strategy. The same gradient-dot-unit-vector formula applies in three dimensions.

Step 1: Direction

$$\overrightarrow{PQ} = \langle -3, 2, 2 \rangle, \quad \|\overrightarrow{PQ}\| = \sqrt{17}, \quad u = \frac{1}{\sqrt{17}} \langle -3, 2, 2 \rangle.$$

Step 2: Gradient

$$F_x = e^{yz}, \quad F_y = xze^{yz}, \quad F_z = xye^{yz} + 2z.$$

At $(2, 0, 1)$,

$$\nabla F(P) = \langle 1, 2, 2 \rangle.$$

Step 3: Rate

$$D_u F(P) = \frac{-3 + 4 + 4}{\sqrt{17}} = \boxed{\frac{5}{\sqrt{17}}}.$$

Because u is dimensionless and unit length, the result measures change in F per unit spatial distance.

Question 7

Define

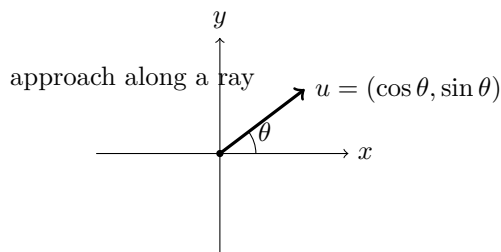
$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $D_u f(0, 0)$ from the definition for every unit $u = \langle \cos \theta, \sin \theta \rangle$.
2. Compare with $\nabla f(0, 0) \cdot u$.
3. Explain what this proves about directional derivatives and differentiability.

Question 7 – Solution

Strategy. Approach the origin along the ray hu and keep the signed parameter h .



Step 1: Definition For $h \neq 0$,

$$f(h \cos \theta, h \sin \theta) = \frac{h^3 \cos^3 \theta}{h^2} = h \cos^3 \theta.$$

Therefore

$$D_u f(0, 0) = \lim_{h \rightarrow 0} \frac{h \cos^3 \theta}{h} = \cos^3 \theta.$$

Step 2: Gradient comparison Axis definitions give $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$, so

$$\nabla f(0, 0) \cdot u = \cos \theta,$$

which differs from $\cos^3 \theta$ for most angles.

Conclusion Every directional derivative exists, but its dependence on u is not linear. Thus f is not differentiable at the origin, and the gradient formula is not valid there.

Question 8

At $P = (1, 2)$, an unknown differentiable function satisfies

$$D_{u_1} f(P) = 4, \quad u_1 = \langle 1, 0 \rangle,$$

$$D_{u_2} f(P) = 1, \quad u_2 = \langle 3/5, 4/5 \rangle.$$

Tasks

1. Reconstruct $\nabla f(P)$.
2. Find the steepest-ascent direction and rate.
3. Predict the derivative in direction $\langle 0, -1 \rangle$ and verify consistency.

Question 8 – Solution

Strategy. Treat directional measurements as linear equations for the two gradient components.

Step 1: Reconstruct Write $\nabla f(P) = \langle a, b \rangle$. The first measurement gives $a = 4$. The second gives

$$\frac{3}{5}a + \frac{4}{5}b = 1.$$

Thus $12/5 + 4b/5 = 1$, so $b = -7/4$ and

$$\boxed{\nabla f(P) = \langle 4, -7/4 \rangle}.$$

Step 2: Maximum

$$\|\nabla f(P)\| = \sqrt{16 + \frac{49}{16}} = \frac{\sqrt{305}}{4}.$$

The steepest-ascent direction is

$$\boxed{\frac{1}{\sqrt{305}} \langle 16, -7 \rangle},$$

and its rate is $\boxed{\sqrt{305}/4}$.

Step 3: New direction For $\langle 0, -1 \rangle$,

$$D_{\langle 0, -1 \rangle} f = \langle 4, -7/4 \rangle \cdot \langle 0, -1 \rangle = \boxed{7/4}.$$

Substitution into both original dot products reproduces 4 and 1.

Question 9

Let $f(x, y, z) = x^2 + y^2 + z^2$. At $P = (1, 2, 2)$, restrict directions to be tangent to the plane $x + y + z = 5$.

Tasks

1. Find the unit tangent direction giving the greatest increase of f .
2. Find that maximum constrained rate.
3. Identify the exceptional case in which no preferred tangent direction exists.

Question 9 – Solution

Strategy. Project the gradient onto the plane's tangent subspace, then normalize the projection.

Step 1: Data $\nabla f(P) = \langle 2, 4, 4 \rangle$. A unit normal to the plane is

$$n = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle.$$

The tangential projection is

$$p = \nabla f - (\nabla f \cdot n)n = \langle 2, 4, 4 \rangle - \frac{10}{3} \langle 1, 1, 1 \rangle = \frac{1}{3} \langle -4, 2, 2 \rangle.$$

Step 2: Direction and rate Since $\|p\| = 2\sqrt{6}/3$,

$$\boxed{u_{\max} = \frac{1}{\sqrt{6}} \langle -2, 1, 1 \rangle}, \quad \boxed{D_{\max, \text{tangent}} f = \frac{2\sqrt{6}}{3}}.$$

Step 3: Verify $u_{\max} \cdot \langle 1, 1, 1 \rangle = 0$, so it is tangent, and $\nabla f \cdot u_{\max} = 2\sqrt{6}/3$. If the projected gradient were zero, every tangent directional derivative would be zero and no unique maximizing tangent direction would exist.

Question 10

At a point P , a differentiable function has unknown gradient. Its directional derivative equals 3 in every unit direction making angle $\pi/3$ with a fixed unit vector a .

Tasks

1. Determine what this implies about the component of $\nabla f(P)$ along a .
2. In two dimensions, use the two such directions to prove the perpendicular component is zero.
3. Determine $\nabla f(P)$ and the maximum directional derivative.

Question 10 – Solution

Strategy. Resolve the gradient into components parallel and perpendicular to a and compare symmetric directions.

Step 1: Decompose Let b be a unit vector perpendicular to a , and write

$$\nabla f = Aa + Bb.$$

The two unit directions at angles $\pm\pi/3$ from a are

$$u_{\pm} = \frac{1}{2}a \pm \frac{\sqrt{3}}{2}b.$$

Step 2: Use both measurements

$$3 = \nabla f \cdot u_{\pm} = \frac{A}{2} \pm \frac{\sqrt{3}B}{2}.$$

Adding gives $6 = A$, while subtracting gives $B = 0$.

Step 3: Result Therefore

$$\boxed{\nabla f(P) = 6a}.$$

The greatest directional derivative is the gradient magnitude,

$$\boxed{D_{\max}f(P) = 6},$$

attained uniquely in direction a . Checking either prescribed direction gives $6 \cos(\pi/3) = 3$.