

Question 1

Let $z = x^2y + \sin y$, where $x = t^2 - 1$ and $y = e^t$.

Tasks

1. Find dz/dt using the multivariable chain rule.
2. Evaluate it at $t = 0$.
3. Verify by differentiating the explicit composite function.

Question 1 – Solution

Strategy. Combine the two routes $t \rightarrow x \rightarrow z$ and $t \rightarrow y \rightarrow z$.

Step 1: Ingredients

$$z_x = 2xy, \quad z_y = x^2 + \cos y, \quad x' = 2t, \quad y' = e^t.$$

Therefore

$$\frac{dz}{dt} = 2xy(2t) + (x^2 + \cos y)e^t.$$

Step 2: Evaluate At $t = 0$, $x = -1$ and $y = 1$, giving

$$\left. \frac{dz}{dt} \right|_{t=0} = 1 + \cos 1.$$

Step 3: Direct check The composite is

$$z(t) = (t^2 - 1)^2 e^t + \sin(e^t).$$

Its derivative is $4t(t^2 - 1)e^t + (t^2 - 1)^2 e^t + \cos(e^t)e^t$, which at zero is $1 + \cos 1$.

Question 2

Suppose $F(x, y) = \ln(x^2 + y^2)$ and a curve is $x = 3 \cos t$, $y = 3 \sin t$.

Tasks

1. Compute dF/dt by the chain rule.
2. Explain the cancellation geometrically.
3. Verify using the explicit composite.

Question 2 – Solution

Strategy. Evaluate the two chain-rule contributions before simplifying.

Step 1: Chain rule

$$F_x = \frac{2x}{x^2 + y^2}, \quad F_y = \frac{2y}{x^2 + y^2},$$
$$x' = -3 \sin t, \quad y' = 3 \cos t.$$

Thus

$$\frac{dF}{dt} = \frac{2x(-3 \sin t) + 2y(3 \cos t)}{x^2 + y^2} = 0,$$

after substituting $x = 3 \cos t$, $y = 3 \sin t$.

Step 2: Meaning The curve remains on the circle $x^2 + y^2 = 9$, and F depends only on that squared radius. Hence $\boxed{dF/dt = 0}$.

Step 3: Direct check $F(3 \cos t, 3 \sin t) = \ln 9$, whose derivative is zero.

Question 3

Let

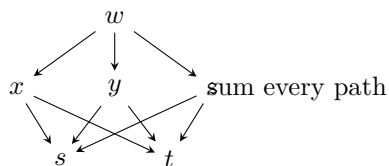
$$w = x^2 + yz, \quad x = s + t, \quad y = st, \quad z = s^2 - t^2.$$

Tasks

1. Draw or use the dependency structure to find w_s, w_t .
2. Evaluate both at $(s, t) = (1, 2)$.
3. Verify one result by explicit expansion.

Question 3 – Solution

Strategy. Sum the contribution along every dependency path.



Step 1: Derivatives

$$w_x = 2x, \quad w_y = z, \quad w_z = y.$$

$$x_s = 1, \quad y_s = t, \quad z_s = 2s; \quad x_t = 1, \quad y_t = s, \quad z_t = -2t.$$

Therefore

$$\boxed{w_s = 2x + zt + 2sy},$$

$$\boxed{w_t = 2x + zs - 2ty}.$$

Step 2: Evaluate At $(1, 2)$, $(x, y, z) = (3, 2, -3)$:

$$\boxed{w_s = 6 - 6 + 4 = 4},$$

$$\boxed{w_t = 6 - 3 - 8 = -5}.$$

Step 3: Check Expanding gives $w = (s+t)^2 + st(s^2 - t^2)$. Differentiating in s yields $2(s+t) + t(3s^2 - t^2)$, which equals 4 at $(1, 2)$.

Question 4

A surface satisfies $x^2 + y^2 + z^2 = 14$. Along a path on the surface, $x = t$, $y = t^2$, and $z > 0$ near $t = 1$.

Tasks

1. Find dz/dt implicitly without solving for $z(t)$ first.
2. Evaluate at $t = 1$.
3. Verify using the positive explicit branch.

Question 4 – Solution

Strategy. Differentiate the constraint along the path; its total rate must remain zero.

Step 1: Differentiate

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0,$$

so

$$\boxed{\frac{dz}{dt} = -\frac{xx' + yy'}{z}}.$$

Step 2: Evaluate At $t = 1$, $x = 1$, $y = 1$, $z = \sqrt{14 - 1 - 1} = 2\sqrt{3}$, while $x' = 1$, $y' = 2$. Hence

$$\boxed{\left. \frac{dz}{dt} \right|_1 = -\frac{3}{2\sqrt{3}} = -\frac{\sqrt{3}}{2}}.$$

Step 3: Check The positive branch is $z(t) = \sqrt{14 - t^2 - t^4}$. Differentiation gives $z' = (-2t - 4t^3)/(2z)$, which produces the same value at $t = 1$.

Question 5

Let $u(x, y)$ be differentiable and set $x = r \cos \theta$, $y = r \sin \theta$.

Tasks

1. Derive formulas for u_r and u_θ .
2. Apply them to $u = x^2 - y^2$.
3. Verify against the polar form of u .

Question 5 – Solution

Strategy. Treat r and θ as independent inputs to the Cartesian coordinate map.

Step 1: General formulas

$$\boxed{u_r = u_x \cos \theta + u_y \sin \theta},$$

$$\boxed{u_\theta = -ru_x \sin \theta + ru_y \cos \theta}.$$

Step 2: Apply For $u = x^2 - y^2$, $u_x = 2x$, $u_y = -2y$. Substituting $x = r \cos \theta$, $y = r \sin \theta$ gives

$$u_r = 2r(\cos^2 \theta - \sin^2 \theta) = \boxed{2r \cos 2\theta},$$

$$u_\theta = -4r^2 \sin \theta \cos \theta = \boxed{-2r^2 \sin 2\theta}.$$

Step 3: Verify Directly, $u = r^2 \cos 2\theta$; its two ordinary partial derivatives match both formulas.

Question 6

Let

$$F(x, y) = \begin{pmatrix} x^2 - y \\ xy \end{pmatrix}, \quad G(s, t) = \begin{pmatrix} e^s \cos t \\ e^s \sin t \end{pmatrix}.$$

Tasks

1. Find the Jacobian matrices J_F and J_G .
2. Use matrix chain rule to find $J_{F \circ G}(0, \pi/2)$.
3. Check one entry directly.

Question 6 – Solution

Strategy. Evaluate J_F at $G(s, t)$ before multiplying by J_G .

Step 1: Jacobians

$$J_F = \begin{pmatrix} 2x & -1 \\ y & x \end{pmatrix}, \quad J_G = \begin{pmatrix} e^s \cos t & -e^s \sin t \\ e^s \sin t & e^s \cos t \end{pmatrix}.$$

At $(0, \pi/2)$, $G = (0, 1)$, so

$$J_F(0, 1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Step 2: Multiply

$$J_{F \circ G}(0, \pi/2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The composite Jacobian is $-I_2$.

Step 3: Check The first component is $e^{2s} \cos^2 t - e^s \sin t$; its s -derivative at the point is $0 - 1 = -1$, matching the upper-left entry.

Question 7

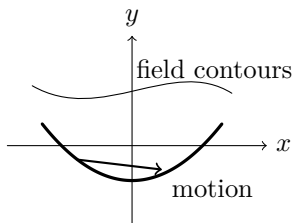
A temperature field is $T(x, y, t) = x^2 + 2xy + 3t$. A particle follows $x = t^2$, $y = 1 - t$.

Tasks

1. Find the rate of temperature change experienced by the particle.
2. Evaluate it at $t = 1$.
3. Separate spatial-motion and explicit-time contributions.

Question 7 – Solution

Strategy. The experienced rate includes motion through space and direct evolution of the field.



Step 1: Extended chain rule

$$\frac{dT}{dt} = T_x x' + T_y y' + T_t.$$

Here

$$T_x = 2x + 2y, \quad T_y = 2x, \quad T_t = 3, \quad x' = 2t, \quad y' = -1.$$

Thus

$$\boxed{\frac{dT}{dt} = (2x + 2y)(2t) - 2x + 3}.$$

Step 2: Evaluate At $t = 1$, $(x, y) = (1, 0)$:

$$\boxed{\frac{dT}{dt} = 2(2) - 2 + 3 = 5}.$$

Step 3: Contributions Motion in x contributes 4, motion in y contributes -2 , and explicit time dependence contributes 3. Their sum is 5 temperature units per time unit.

Question 8

Suppose $z = f(x, y)$ has $f_x(2, 0) = 3$ and $f_y(2, 0) = -4$. The inputs obey

$$x = u^2 + v, \quad y = u - v^2.$$

At $(u, v) = (1, 1)$:

Tasks

1. Find z_u and z_v .
2. Find all input changes (du, dv) producing zero first-order change in z .
3. Verify the cancellation using dx, dy .

Question 8 – Solution

Strategy. First propagate u, v changes to x, y , then combine them with the supplied partials.

Step 1: Chain derivatives At $(1, 1)$, $(x, y) = (2, 0)$ and

$$x_u = 2, \ x_v = 1, \ y_u = 1, \ y_v = -2.$$

Therefore

$$\boxed{z_u = 3(2) + (-4)(1) = 2}, \quad \boxed{z_v = 3(1) + (-4)(-2) = 11}.$$

Step 2: Zero-change inputs The total differential is

$$dz = 2 \, du + 11 \, dv.$$

Thus every first-order neutral change satisfies

$$\boxed{2 \, du + 11 \, dv = 0},$$

equivalently $(du, dv) = \lambda(11, -2)$.

Step 3: Verify For $(du, dv) = (11, -2)$, $dx = 2(11) - 2 = 20$ and $dy = 11 - 2(-2) = 15$. Then $dz = 3(20) - 4(15) = 0$.

Question 9

Variables x, y are defined implicitly by

$$x + y = s, \quad xy = t.$$

Near $(x, y) = (1, 2)$, regard x, y as functions of (s, t) .

Tasks

1. Find x_s, y_s, x_t, y_t by differentiating the system.
2. Evaluate them at $(1, 2)$.
3. Explain why the calculation fails when $x = y$.

Question 9 – Solution

Strategy. Differentiate both constraints and solve the resulting 2×2 linear systems.

Step 1: With respect to s

$$x_s + y_s = 1, \quad yx_s + xy_s = 0.$$

Solving gives

$$\boxed{x_s = \frac{x}{x-y}}, \quad \boxed{y_s = -\frac{y}{x-y}}.$$

Step 2: With respect to t

$$x_t + y_t = 0, \quad yx_t + xy_t = 1,$$

so

$$\boxed{x_t = \frac{1}{y-x}}, \quad \boxed{y_t = \frac{1}{x-y}}.$$

At $(1, 2)$:

$$\boxed{x_s = -1, \ y_s = 2, \ x_t = 1, \ y_t = -1}.$$

Step 3: Edge case The determinant of the coefficient matrix is $x - y$. When $x = y$, the system cannot locally distinguish the two roots; division by $x - y$ correctly signals failure of this local inverse description.

Question 10

Let $z = f(x(t), y(t))$, where f has continuous second partials.

Tasks

1. Derive a formula for d^2z/dt^2 .
2. Organize it into curvature-of- f and acceleration-of-path terms.
3. Apply it to $f = x^2 + xy + y^2$, $x = \cos t$, $y = \sin t$ at $t = 0$.

Question 10 – Solution

Strategy. Differentiate $z' = f_x x' + f_y y'$ and apply the chain and product rules to every factor.

Step 1: General formula

$$z'' = f_{xx}(x')^2 + 2f_{xy}x'y' + f_{yy}(y')^2 + f_x x'' + f_y y''.$$

The first three terms measure the function's second-order response along the velocity; the last two measure the effect of path acceleration on the first partials.

Step 2: Data For $f = x^2 + xy + y^2$,

$$f_x = 2x + y, \quad f_y = x + 2y, \quad f_{xx} = 2, \quad f_{xy} = 1, \quad f_{yy} = 2.$$

At $t = 0$, $(x, y) = (1, 0)$, $(x', y') = (0, 1)$, and $(x'', y'') = (-1, 0)$.

Step 3: Evaluate

$$z'' = 2(0)^2 + 2(1)(0)(1) + 2(1)^2 + (2)(-1) + (1)(0) = \boxed{0}.$$

Directly, $z = \cos^2 t + \sin t \cos t + \sin^2 t = 1 + \frac{1}{2} \sin 2t$, whose second derivative is $-\sin 2t$, also zero at $t = 0$.