

Question 1

For

$$z = x^2 e^y + \ln(x + y), \quad x + y > 0,$$

Tasks

1. Find the total differential dz .
2. Evaluate it at $(1, 1)$ for $dx = -0.02$, $dy = 0.03$.
3. Identify the contribution from each input change.

Question 1 – Solution

Strategy. Use $dz = f_x dx + f_y dy$ and evaluate coefficients before applying the increments.

Step 1: Partial derivatives

$$f_x = 2xe^y + \frac{1}{x+y}, \quad f_y = x^2e^y + \frac{1}{x+y}.$$

Thus

$$dz = \left(2xe^y + \frac{1}{x+y} \right) dx + \left(x^2e^y + \frac{1}{x+y} \right) dy.$$

Step 2: Evaluate At $(1, 1)$,

$$dz = (2e + \tfrac{1}{2})(-0.02) + (e + \tfrac{1}{2})(0.03).$$

Combining terms,

$$dz = -0.04e - 0.01 + 0.03e + 0.015 = \boxed{0.005 - 0.01e \approx -0.02218}.$$

Step 3: Contributions The x change contributes $-(0.04e + 0.01)$; the y change contributes $0.03e + 0.015$. Their signs oppose, and the x contribution has larger magnitude.

Question 2

Use differentials to estimate

$$(4.02)^2\sqrt{8.97}.$$

Tasks

1. Choose a two-variable function and nearby convenient base point.
2. Compute the differential estimate.
3. State the estimated value and identify the neglected order of error.

Question 2 – Solution

Strategy. Let $f(x, y) = x^2\sqrt{y}$ and linearize its change from $(4, 9)$.

Step 1: Base data

$$f(4, 9) = 16(3) = 48, \quad dx = 0.02, \quad dy = -0.03.$$

$$f_x = 2x\sqrt{y}, \quad f_y = \frac{x^2}{2\sqrt{y}}.$$

At $(4, 9)$, $f_x = 24$ and $f_y = 8/3$.

Step 2: Differential

$$df = 24(0.02) + \frac{8}{3}(-0.03) = 0.48 - 0.08 = 0.40.$$

Step 3: Estimate

$$\boxed{(4.02)^2\sqrt{8.97} \approx 48 + 0.40 = 48.40}.$$

The differential keeps terms linear in dx, dy and neglects quadratic and higher-order increment effects.

Question 3

For $f(x, y) = x^2 + y^2$, start at $(3, 4)$ and use $dx = 0.1$, $dy = -0.2$.

Tasks

1. Compute df .
2. Compute the exact change Δf .
3. Derive the discrepancy $\Delta f - df$ and explain its sign.

Question 3 – Solution

Strategy. Expand the exact finite change so the first-order and second-order parts separate visibly.

Step 1: Differential

$$df = 2x \, dx + 2y \, dy = 6(0.1) + 8(-0.2) = \boxed{-1.0}.$$

Step 2: Exact change

$$\Delta f = (3.1)^2 + (3.8)^2 - (3^2 + 4^2) = 9.61 + 14.44 - 25 = \boxed{-0.95}.$$

Step 3: Discrepancy Algebraically,

$$\Delta f = 2x \, dx + 2y \, dy + (dx)^2 + (dy)^2 = df + (dx)^2 + (dy)^2.$$

Therefore

$$\boxed{\Delta f - df = 0.01 + 0.04 = 0.05}.$$

It is nonnegative because the omitted terms are squares; the upward curvature makes the exact change exceed the linear prediction.

Question 4

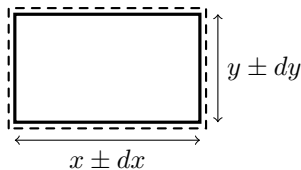
A rectangular plate has measured sides $x = 30.0$ cm and $y = 20.0$ cm, each accurate to within 0.1 cm. Its area is $A = xy$.

Tasks

1. Use differentials to estimate the maximum absolute area error.
2. Estimate the maximum relative and percentage errors.
3. Compare with the exact worst-case outward error.

Question 4 – Solution

Strategy. For a worst-case differential bound, replace signed increments by their maximum absolute values.



Step 1: Absolute error

$$dA = y \, dx + x \, dy.$$

Thus

$$|dA| \leq 20(0.1) + 30(0.1) = \boxed{5.0 \text{ cm}^2}.$$

Step 2: Relative error Since $A = 600 \text{ cm}^2$,

$$\frac{|dA|}{A} \approx \frac{5}{600} = \boxed{\frac{1}{120} \approx 0.00833},$$

or $\boxed{0.833\%}$.

Step 3: Exact comparison The largest outward area is

$$(30.1)(20.1) = 605.01,$$

so the exact positive error is 5.01 cm^2 . The differential misses only the product $dx \, dy = 0.01 \text{ cm}^2$.

Question 5

A cone has volume $V(r, h) = \frac{1}{3}\pi r^2 h$. At $r = 5$ cm, $h = 12$ cm, suppose $dr = 0.04$ cm and $dh = -0.10$ cm.

Tasks

1. Find dV exactly in terms of π .
2. Determine which measurement change dominates the estimate.
3. Compute the relative differential dV/V in two ways.

Question 5 – Solution

Strategy. Compute component contributions, then confirm the relative formula by logarithmic structure.

Step 1: Differential

$$dV = \frac{2\pi r h}{3} dr + \frac{\pi r^2}{3} dh.$$

At the given dimensions,

$$dV = 40\pi(0.04) + \frac{25\pi}{3}(-0.10) = 1.6\pi - \frac{5}{6}\pi = \boxed{\frac{23\pi}{30} \text{ cm}^3}.$$

Step 2: Compare The radial contribution is 1.6π ; the height contribution is $-5\pi/6$. The radial contribution dominates, so the net predicted change is positive.

Step 3: Relative form

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h} = 2\frac{0.04}{5} - \frac{0.10}{12} = \boxed{\frac{23}{3000}}.$$

Also $V = 100\pi$, and $(23\pi/30)/(100\pi) = 23/3000$, verifying the result.

Question 6

Points constrained to the sphere

$$x^2 + y^2 + z^2 = 49$$

undergo small changes near $(2, 3, 6)$.

Tasks

1. Derive the differential constraint relating dx, dy, dz .
2. Estimate dz when $dx = 0.03$ and $dy = -0.02$.
3. Explain geometrically why the first-order change satisfies this relation.

Question 6 – Solution

Strategy. Differentiate the constant constraint; its total differential must vanish along allowed infinitesimal changes.

Step 1: Constraint

$$2x \, dx + 2y \, dy + 2z \, dz = 0,$$

so, because $z = 6 \neq 0$,

$$\boxed{dz = -\frac{x \, dx + y \, dy}{z}}.$$

Step 2: Estimate

$$dz = -\frac{2(0.03) + 3(-0.02)}{6} = -\frac{0.06 - 0.06}{6} = \boxed{0}.$$

Step 3: Geometry The radius vector $\langle x, y, z \rangle$ is normal to the sphere. The relation

$$\langle x, y, z \rangle \cdot \langle dx, dy, dz \rangle = 0$$

says the first-order displacement is perpendicular to that normal, hence tangent to the sphere. Here the chosen horizontal change is already orthogonal to the radius projection, so no first-order vertical correction is needed.

Question 7

Determine whether

$$\omega = (2xy + 3) dx + (x^2 + 4y) dy$$

is the differential of a scalar function $f(x, y)$.

Tasks

1. Test the necessary compatibility condition.
2. Recover every possible f if the test passes.
3. Verify by redifferentiating.

Question 7 – Solution

Strategy. If $df = M dx + N dy$, then $M_y = N_x$ on a simply connected domain.

Step 1: Compatibility Here

$$M = 2xy + 3, \quad N = x^2 + 4y.$$

Then

$$M_y = 2x = N_x,$$

so the form is compatible throughout $\mathbb{R}^2$.

Step 2: Recover f Integrate $f_x = M$ in x :

$$f = x^2y + 3x + C(y).$$

Then

$$f_y = x^2 + C'(y) = x^2 + 4y,$$

so $C'(y) = 4y$ and $C(y) = 2y^2 + C_0$.

Result and verification

$$\boxed{f(x, y) = x^2y + 3x + 2y^2 + C_0}.$$

Indeed, differentiating gives $df = (2xy + 3) dx + (x^2 + 4y) dy$. The additive constant is invisible to the differential.

Question 8

A quantity is

$$Q = \frac{x^3 \sqrt{y}}{z^2}, \quad x, y, z > 0.$$

Tasks

1. Derive a relative-differential formula for dQ/Q .
2. Estimate the signed percentage change if x rises 1%, y falls 4%, and z rises 0.5%.
3. Identify the dominant contribution.

Question 8 – Solution

Strategy. Divide the total differential by Q , or equivalently differentiate the logarithmic power structure.

Step 1: Relative differential

$$\ln Q = 3 \ln x + \frac{1}{2} \ln y - 2 \ln z.$$

Differentiating gives

$$\boxed{\frac{dQ}{Q} = 3 \frac{dx}{x} + \frac{1}{2} \frac{dy}{y} - 2 \frac{dz}{z}}.$$

Step 2: Percentage estimate

$$\frac{dQ}{Q} \approx 3(0.01) + \frac{1}{2}(-0.04) - 2(0.005) = 0.03 - 0.02 - 0.01 = \boxed{0}.$$

Thus the predicted first-order percentage change is $\boxed{0\%}$.

Step 3: Contributions The x increase contributes $+3\%$, the y decrease -2% , and the z increase -1% . The x term is individually largest, but the three first-order effects cancel exactly.

Question 9

A differential field is proposed as

$$df = (y^2 + 2x) dx + (2xy + e^y) dy, \quad f(0, 0) = 5.$$

Tasks

1. Reconstruct f .
2. Use df to estimate $f(0.02, -0.01)$ from the origin.
3. Compare the estimate with the exact value through second order.

Question 9 – Solution

Strategy. Reconstruct the function first, then separate its linear and nonlinear increment terms.

Step 1: Reconstruct Integrating $f_x = y^2 + 2x$ gives

$$f = xy^2 + x^2 + C(y).$$

Since $f_y = 2xy + C'(y) = 2xy + e^y$, $C = e^y + C_0$. The initial value gives $1 + C_0 = 5$, so

$$\boxed{f = xy^2 + x^2 + e^y + 4}.$$

Step 2: Differential estimate At $(0, 0)$,

$$f_x = 0, \quad f_y = 1.$$

For $dx = 0.02$, $dy = -0.01$,

$$f(0.02, -0.01) \approx 5 + df = 5 - 0.01 = \boxed{4.99}.$$

Step 3: Exact comparison Exactly,

$$f = 0.02(0.01)^2 + (0.02)^2 + e^{-0.01} + 4 \approx 4.99045183.$$

The estimate error is about $\boxed{0.00045183}$, while the second-order correction is $dx^2 + dy^2/2 = 0.00045$. Thus the second-order estimate is 4.99045, consistent with the exact value.

Question 10

Define

$$f(x, y) = \begin{cases} \frac{x^3}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $f_x(0, 0)$ and $f_y(0, 0)$.
2. Test whether a differential $df = f_x dx + f_y dy$ gives a valid first-order approximation at the origin.
3. Identify a path that proves or disproves differentiability.

Question 10 – Solution

Strategy. Existing partial derivatives propose a linear map, but differentiability requires the remainder divided by distance to vanish along every path.

Step 1: Partial

$$f_x(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0)}{h} = \lim_{h \rightarrow 0} \frac{h}{h} = 1,$$

$$f_y(0,0) = \lim_{k \rightarrow 0} \frac{f(0,k)}{k} = 0.$$

The candidate differential is therefore $L(dx, dy) = dx$.

Step 2: Remainder test Differentiability would require

$$\frac{f(x,y) - x}{\sqrt{x^2 + y^2}} \rightarrow 0.$$

Along $y = x$,

$$f(x,x) = \frac{x^3}{2x^2} = \frac{x}{2},$$

so

$$\frac{f(x,x) - x}{\sqrt{2x^2}} = \frac{-x/2}{\sqrt{2}|x|} = -\frac{\text{sgn}(x)}{2\sqrt{2}},$$

which does not tend to zero.

Conclusion Although both partials exist, f is not differentiable at the origin, so

$df = dx$ is not a valid total first-order approximation there.