

**Question 1**

For

$$f(x, y) = x^4y^2 - 3x^2y^3 + 2xy,$$

**Tasks**

1. Find  $f_{xx}, f_{xy}, f_{yx}, f_{yy}$ .
2. Evaluate them at  $(1, -1)$ .
3. Verify the mixed-partial equality algebraically.

## Question 1 – Solution

**Strategy.** Compute both first partials before differentiating again.

**Step 1: First partials**

$$f_x = 4x^3y^2 - 6xy^3 + 2y, \quad f_y = 2x^4y - 9x^2y^2 + 2x.$$

**Step 2: Second partials**

$$\boxed{f_{xx} = 12x^2y^2 - 6y^3}, \quad \boxed{f_{xy} = 8x^3y - 18xy^2 + 2},$$
$$\boxed{f_{yx} = 8x^3y - 18xy^2 + 2}, \quad \boxed{f_{yy} = 2x^4 - 18x^2y}.$$

**Step 3: Evaluate**

$$\boxed{f_{xx}(1, -1) = 18}, \quad \boxed{f_{xy}(1, -1) = f_{yx}(1, -1) = -24}, \quad \boxed{f_{yy}(1, -1) = 20}.$$

The two mixed formulas coincide term by term, as expected for a polynomial.

## Question 2

Let  $g(x, y) = e^{xy} \cos y$ .

### Tasks

1. Compute  $g_{xx}$  and  $g_{xy}$ .
2. Compute  $g_{yx}$  independently from  $g_y$ .
3. Evaluate them at  $(0, \pi)$  and verify agreement.

**Question 2 – Solution**

**Strategy.** Keep product-rule terms grouped by their exponential factor.

**Step 1: Differentiate from  $g_x$**  Since  $g_x = ye^{xy} \cos y$ ,

$$\boxed{g_{xx} = y^2 e^{xy} \cos y},$$

$$\boxed{g_{xy} = e^{xy} [(1 + xy) \cos y - y \sin y]}.$$

**Step 2: Reverse order**

$$g_y = e^{xy} (x \cos y - \sin y),$$

$$g_{yx} = ye^{xy} (x \cos y - \sin y) + e^{xy} \cos y = \boxed{e^{xy} [(1 + xy) \cos y - y \sin y]}.$$

**Step 3: Evaluate** At  $(0, \pi)$ ,  $e^{xy} = 1$ ,  $\cos \pi = -1$ , and  $\sin \pi = 0$ :

$$\boxed{g_{xx} = -\pi^2}, \quad \boxed{g_{xy} = g_{yx} = -1}.$$

**Question 3**

For  $u(x, y) = \ln(x^2 + y^2)$  on its natural domain:

**Tasks**

1. Find all second partial derivatives.
2. Show that  $u_{xx} + u_{yy} = 0$ .
3. Explain why the result cannot be extended to the origin by substitution.

**Question 3 – Solution**

**Strategy.** Differentiate the rational first partials and preserve the punctured-plane domain.

**Step 1: First partials**

$$u_x = \frac{2x}{x^2 + y^2}, \quad u_y = \frac{2y}{x^2 + y^2}.$$

**Step 2: Second partials**

$$\boxed{u_{xx} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}}, \quad \boxed{u_{yy} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}},$$
$$\boxed{u_{xy} = u_{yx} = -\frac{4xy}{(x^2 + y^2)^2}}.$$

**Step 3: Cancellation**

$$u_{xx} + u_{yy} = \frac{2(y^2 - x^2 + x^2 - y^2)}{(x^2 + y^2)^2} = \boxed{0}.$$

The origin is excluded from  $u$ , and every denominator above vanishes there; substitution cannot create a derivative at a point outside the domain.

**Question 4**

Find the third-order derivative  $f_{xxy}$  for

$$f(x, y) = x^2 e^{xy}.$$

**Tasks**

1. Compute it in the encoded order.
2. Compute  $f_{yxx}$  independently.
3. Explain the agreement and evaluate at  $(1, 0)$ .

**Question 4 – Solution**

**Strategy.** Follow each subscript successively and retain all product-rule terms.

**Step 1: Route**  $x, x, y$

$$\begin{aligned}f_x &= e^{xy}(2x + x^2y), \\f_{xx} &= e^{xy}(2 + 4xy + x^2y^2), \\f_{xxy} &= e^{xy}(6x + 6x^2y + x^3y^2).\end{aligned}$$

**Step 2: Route**  $y, x, x$

$$\begin{aligned}f_y &= x^3e^{xy}, \quad f_{yx} = e^{xy}(3x^2 + x^3y), \\f_{yxx} &= e^{xy}(6x + 6x^2y + x^3y^2).\end{aligned}$$

**Step 3: Verify** Polynomial and exponential factors are smooth, so the derivative order may be permuted. At  $(1, 0)$  both expressions equal  $\boxed{6}$ .

**Question 5**

A proposed Hessian is

$$H_f(x, y) = \begin{pmatrix} 6x - 2y & -2x + 4y \\ -2x + 4y & 4x + 6y \end{pmatrix}.$$

**Tasks**

1. Check its compatibility conditions.
2. Recover one possible  $f(x, y)$ .
3. Describe every function with this Hessian.

## Question 5 – Solution

**Strategy.** Integrate  $f_{xx}$  and use the other entries to determine the unknown functions of integration.

**Step 1: Compatibility** The off-diagonal entries agree, and

$$\partial_y f_{xx} = -2 = \partial_x f_{xy}, \quad \partial_y f_{xy} = 4 = \partial_x f_{yy}.$$

**Step 2: Reconstruct**

$$f_x = 3x^2 - 2xy + A(y).$$

Since  $f_{xy} = -2x + A'(y) = -2x + 4y$ ,  $A(y) = 2y^2 + C$ . Integrating in  $x$ ,

$$f = x^3 - x^2y + 2xy^2 + Cx + B(y).$$

The condition  $f_{yy} = 4x + B''(y) = 4x + 6y$  gives  $B = y^3 + Dy + E$ .

**Result**

$$\boxed{f = x^3 - x^2y + 2xy^2 + y^3 + Cx + Dy + E}.$$

Conversely, direct differentiation shows every choice of  $C, D, E$  has exactly the proposed Hessian.

**Question 6**

Assume the mixed partials of  $f$  are continuous near  $(a, b)$ . Define

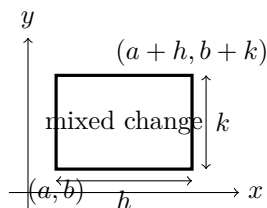
$$D(h, k) = \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk}.$$

**Tasks**

1. Interpret the numerator as a rectangular mixed change.
2. Apply the mean value theorem in each variable.
3. Find  $\lim_{(h,k) \rightarrow (0,0)} D(h, k)$ .

## Question 6 – Solution

**Strategy.** Apply one-variable mean value arguments successively across the small rectangle.



**Step 1: Difference structure** Let

$$G(y) = f(a + h, y) - f(a, y).$$

The numerator is  $G(b + k) - G(b)$ .

**Step 2: Mean value steps** For some  $\eta$  between  $b$  and  $b + k$ ,

$$G(b + k) - G(b) = kG'(\eta) = k[f_y(a + h, \eta) - f_y(a, \eta)].$$

For some  $\xi$  between  $a$  and  $a + h$ , the bracket equals  $hf_{yx}(\xi, \eta)$ . Hence

$$D(h, k) = f_{yx}(\xi, \eta).$$

**Step 3: Limit** As  $(h, k) \rightarrow (0, 0)$ ,  $(\xi, \eta) \rightarrow (a, b)$ . Continuity and equality of mixed partials give

$$\boxed{\lim D(h, k) = f_{yx}(a, b) = f_{xy}(a, b)}.$$

**Question 7**

For  $w(x, y, z) = x^2yz^3 + \sin(xz)$ :

**Tasks**

1. Find  $w_{xyz}$ .
2. Find  $w_{zzx}$ .
3. Evaluate both at  $(1, 2, 0)$  and discuss derivative order.

**Question 7 – Solution**

**Strategy.** Follow the subscripts and keep polynomial and trigonometric contributions separate.

**Step 1:**  $w_{xyz}$

$$w_x = 2xyz^3 + z \cos(xz), \quad w_{xy} = 2xz^3, \quad \boxed{w_{xyz} = 6xz^2}.$$

**Step 2:**  $w_{zzx}$

$$\begin{aligned} w_z &= 3x^2yz^2 + x \cos(xz), \\ w_{zz} &= 6x^2yz - x^2 \sin(xz), \\ \boxed{w_{zzx} &= 12xyz - 2x \sin(xz) - x^2 z \cos(xz)}. \end{aligned}$$

**Step 3: Evaluate** Both happen to equal  $\boxed{0}$  at  $(1, 2, 0)$ . Smoothness permits reordering derivatives that contain the same multiset of variables;  $xyz$  and  $zzx$  are different derivative types and need not agree generally.

**Question 8**

Let  $f(x, y) = x^2y^2/(x^2 + y^2)$  away from  $(0, 0)$  and  $f(0, 0) = 0$ .

**Tasks**

1. Find  $f_x(0, y)$  and  $f_y(x, 0)$  from definitions.
2. Compute  $f_{xy}(0, 0)$  and  $f_{yx}(0, 0)$ .
3. Explain why axis data suffice here.

## Question 8 – Solution

**Strategy.** Restrict first derivatives to an axis, then differentiate that restriction at zero.

**Step 1: Axis first partials** For  $y \neq 0$ ,

$$f_x(0, y) = \lim_{h \rightarrow 0} \frac{h^2 y^2 / (h^2 + y^2)}{h} = \lim_{h \rightarrow 0} \frac{h y^2}{h^2 + y^2} = 0.$$

At  $y = 0$  it is also zero. By symmetry,  $f_y(x, 0) = 0$  for every  $x$ .

**Step 2: Mixed values**

$$f_{xy}(0, 0) = \lim_{k \rightarrow 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = 0,$$

$$f_{yx}(0, 0) = \lim_{h \rightarrow 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = 0.$$

Thus  $\boxed{f_{xy}(0, 0) = f_{yx}(0, 0) = 0}$ .

**Verification.** Each definition uses only the relevant axis restriction, so off-axis values do not enter these two particular quotients.

**Question 9**

Suppose  $u$  satisfies  $u_{tt} = 9u_{xx}$  and

$$u(x, t) = A \cos(2x) \cos(\omega t).$$

**Tasks**

1. Compute  $u_{tt}$  and  $u_{xx}$ .
2. Determine all real  $\omega$  for nonzero  $A$ .
3. Verify the result and analyze the  $A = 0$  edge case.

## Question 9 – Solution

**Strategy.** Substitute the separated trial form and compare its common factors.

**Step 1: Derivatives**

$$u_{tt} = -A\omega^2 \cos(2x) \cos(\omega t), \quad u_{xx} = -4A \cos(2x) \cos(\omega t).$$

**Step 2: Match** For  $A \neq 0$ , equality for every  $(x, t)$  requires

$$-\omega^2 = 9(-4), \quad \omega^2 = 36,$$

so  $\boxed{\omega = \pm 6}$ .

**Step 3: Verify** Directly,  $u_{tt} = -36u = 9(-4u) = 9u_{xx}$ . Because  $\cos(-6t) = \cos(6t)$ , the signs produce the same function. If  $A = 0$ , the zero function satisfies the equation for every  $\omega$ .

**Question 10**

Find all real constants  $a, b, c$  for which

$$u(x, y) = e^{ax+by} \cos(cx)$$

satisfies  $u_{xx} + u_{yy} = 0$  everywhere.

**Tasks**

1. Compute the pure second partials.
2. Separate sine and cosine coefficients.
3. Solve all conditions, including  $c = 0$ .

## Question 10 – Solution

**Strategy.** Equality for every  $x$  forces independent trigonometric coefficients to vanish.

**Step 1: Differentiate**

$$\begin{aligned}u_{xx} &= e^{ax+by}[(a^2 - c^2)\cos(cx) - 2ac\sin(cx)], \\u_{yy} &= b^2 e^{ax+by}\cos(cx).\end{aligned}$$

Thus the sum is

$$e^{ax+by}[(a^2 + b^2 - c^2)\cos(cx) - 2ac\sin(cx)].$$

**Step 2: Nonzero  $c$**  If  $c \neq 0$ , independence of sine and cosine gives

$$a^2 + b^2 - c^2 = 0, \quad ac = 0.$$

Thus  $a = 0$  and  $b = \pm c$ .

**Step 3: Edge case** If  $c = 0$ , the condition is  $a^2 + b^2 = 0$ , hence  $a = b = 0$ . This is already included below:

$$\boxed{(a, b, c) = (0, c, c) \text{ or } (0, -c, c), \quad c \in \mathbb{R}}.$$

Substitution makes both coefficients vanish, verifying sufficiency.