

Question 1

Let

$$f(x, y) = 4x^3y^2 - 5x^2y + 7y^3 - 6.$$

Tasks

1. Find f_x and f_y .
2. Evaluate both at $(1, -2)$.
3. Verify that each differentiation treats the other variable as constant.

Question 1 – Solution

Strategy. Differentiate term by term, holding the non-differentiated variable fixed.

Step 1: Differentiate with respect to x

$$f_x = 12x^2y^2 - 10xy.$$

The terms $7y^3$ and -6 are constant with respect to x .

Step 2: Differentiate with respect to y

$$f_y = 8x^3y - 5x^2 + 21y^2.$$

Step 3: Evaluate

$$f_x(1, -2) = 12(1)(4) - 10(1)(-2) = \boxed{68},$$

$$f_y(1, -2) = 8(1)(-2) - 5(1) + 21(4) = \boxed{63}.$$

Verification. The x -derivative lowers only powers of x ; the y -derivative lowers only powers of y . Substitution gives $48 + 20 = 68$ and $-16 - 5 + 84 = 63$.

Question 2

For

$$g(x, y) = \frac{x^2 + 3y}{2x - y},$$

find its first partial derivatives on its domain.

Tasks

1. State the domain restriction.
2. Compute g_x and g_y using the quotient rule.
3. Simplify and check the denominators.

Question 2 – Solution

Strategy. Apply the one-variable quotient rule separately in each variable and preserve the original domain.

Step 1: Domain The function and its partial derivatives are defined where

$$\boxed{2x - y \neq 0}.$$

Step 2: x -partial Holding y fixed,

$$\begin{aligned} g_x &= \frac{(2x)(2x - y) - 2(x^2 + 3y)}{(2x - y)^2} \\ &= \boxed{\frac{2x^2 - 2xy - 6y}{(2x - y)^2}}. \end{aligned}$$

Step 3: y -partial Holding x fixed,

$$\begin{aligned} g_y &= \frac{3(2x - y) - (x^2 + 3y)(-1)}{(2x - y)^2} \\ &= \boxed{\frac{x^2 + 6x}{(2x - y)^2}}. \end{aligned}$$

Verification. Both results have the squared quotient-rule denominator and remain restricted to $2x - y \neq 0$; expanding each unsimplified numerator reproduces the displayed form.

Question 3

Let

$$h(x, y) = e^{xy} \sin(x + y).$$

Tasks

1. Compute h_x and h_y .
2. Evaluate both at $(0, \pi/2)$.
3. Identify the product-rule contribution that vanishes at the point.

Question 3 – Solution

Strategy. Use the product rule, differentiating each composite factor with only one variable active.

Step 1: x -partial

$$h_x = ye^{xy} \sin(x+y) + e^{xy} \cos(x+y) = e^{xy} [y \sin(x+y) + \cos(x+y)].$$

Step 2: y -partial

$$h_y = xe^{xy} \sin(x+y) + e^{xy} \cos(x+y) = e^{xy} [x \sin(x+y) + \cos(x+y)].$$

Step 3: Evaluate At $(0, \pi/2)$, $e^{xy} = 1$, $\sin(x+y) = 1$, and $\cos(x+y) = 0$. Therefore

$$\boxed{h_x(0, \pi/2) = \pi/2}, \quad \boxed{h_y(0, \pi/2) = 0}.$$

The cosine contribution vanishes in both partials; the $xe^{xy} \sin(x+y)$ contribution in h_y also vanishes because $x = 0$.

Question 4

On the domain $x > 0$, define

$$p(x, y) = x^y.$$

Tasks

1. Find p_x by treating y as constant.
2. Find p_y using an exponential representation.
3. Evaluate both partial derivatives at $(e, 2)$ and verify their forms.

Question 4 – Solution

Strategy. Use the ordinary power rule for x and write $x^y = e^{y \ln x}$ when differentiating with respect to the exponent.

Step 1: x -partial With y fixed,

$$\boxed{p_x(x, y) = yx^{y-1}}.$$

Step 2: y -partial Since $p = e^{y \ln x}$ and $\ln x$ is constant with respect to y ,

$$\boxed{p_y(x, y) = x^y \ln x}.$$

Step 3: Evaluate

$$p_x(e, 2) = 2e, \quad p_y(e, 2) = e^2 \ln e = e^2.$$

Thus

$$\boxed{p_x(e, 2) = 2e, \quad p_y(e, 2) = e^2}.$$

Verification. For fixed $y = 2$, $p = x^2$ has derivative $2x$. For fixed $x = e$, $p = e^y$ has derivative e^y , confirming both values.

Question 5

Let

$$F(x, y, z) = x^2yz + \ln(x + z) - ye^z, \quad x + z > 0.$$

Tasks

1. Compute F_x , F_y , and F_z .
2. Evaluate all three at $(1, 2, 0)$.
3. Check which terms are constant in each differentiation.

Question 5 – Solution

Strategy. Differentiate with one variable active at a time and keep the logarithm's domain restriction.

Step 1: Partial derivatives

$$F_x = 2xyz + \frac{1}{x+z},$$

$$F_y = x^2z - e^z,$$

$$F_z = x^2y + \frac{1}{x+z} - ye^z.$$

Step 2: Evaluate At $(1, 2, 0)$,

$$F_x = 0 + 1 = \boxed{1}, \quad F_y = 0 - 1 = \boxed{-1},$$

$$F_z = 2 + 1 - 2 = \boxed{1}.$$

Verification. In F_x , $-ye^z$ is constant; in F_y , $\ln(x+z)$ is constant; in F_z , every term varies with z . The point satisfies $x+z=1>0$.

Question 6

The equation

$$x^2 + y^2 + z^2 + xyz = 4$$

defines z locally as a function of x and y near $P = (1, 1, 1)$.

Tasks

1. Find z_x and z_y by partial implicit differentiation.
2. Evaluate them at P .
3. State the nonzero factor that permits solving for the derivatives.

Question 6 – Solution

Strategy. Differentiate the equation while treating $z = z(x, y)$, then collect the terms containing the requested derivative.

Step 1: Differentiate in x

$$2x + 2zz_x + yz + xyz_x = 0,$$

so

$$z_x = -\frac{2x + yz}{2z + xy}.$$

Step 2: Differentiate in y

$$2y + 2zz_y + xz + xyz_y = 0,$$

so

$$z_y = -\frac{2y + xz}{2z + xy}.$$

Step 3: Evaluate At $(1, 1, 1)$, the common denominator is $2z + xy = 3 \neq 0$. Hence

$$z_x(1, 1) = -1, \quad z_y(1, 1) = -1.$$

Substitution into the differentiated equations gives $2 + 2(-1) + 1 + (-1) = 0$ in each case.

Question 7

For the surface

$$z = f(x, y) = x^2 + xy + y^2,$$

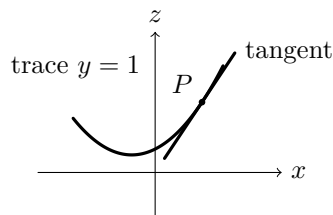
hold $y = 1$ fixed and examine the point $P = (1, 1, 3)$.

Tasks

1. Find $f_x(1, 1)$.
2. Obtain the tangent line to the trace $y = 1$ at P .
3. Verify the slope by differentiating the trace directly.

Question 7 – Solution

Strategy. Restrict the surface to the vertical plane $y = 1$; the resulting one-variable curve has slope $f_x(1, 1)$.



Step 1: Partial derivative

$$f_x = 2x + y, \quad \boxed{f_x(1, 1) = 3}.$$

Step 2: Trace and tangent At $y = 1$,

$$z = x^2 + x + 1.$$

The tangent line in the plane $y = 1$ passes through $(x, z) = (1, 3)$ with slope 3:

$$\boxed{y = 1, \quad z - 3 = 3(x - 1)}.$$

Step 3: Direct check Differentiating the trace gives $dz/dx = 2x + 1$, which equals 3 at $x = 1$, matching the partial derivative.

Question 8

Determine whether constants a , b , and c exist such that

$$q(x, y) = ax^2 + bxy + cy^2$$

satisfies

$$q_x(1, 1) = 7, \quad q_y(1, 1) = 8, \quad q(1, 1) = 6.$$

Tasks

1. Translate the data into a linear system.
2. Solve for a, b, c or prove inconsistency.
3. Verify your conclusion against all three conditions.

Question 8 – Solution

Strategy. Compute the partial derivatives symbolically, evaluate at $(1, 1)$, and solve the resulting three equations.

Step 1: Equations

$$q_x = 2ax + by, \quad q_y = bx + 2cy.$$

Thus

$$2a + b = 7, \quad b + 2c = 8, \quad a + b + c = 6.$$

Step 2: Solve From the first two equations,

$$a = \frac{7-b}{2}, \quad c = \frac{8-b}{2}.$$

Substitution into $a + b + c = 6$ gives

$$\frac{7-b}{2} + b + \frac{8-b}{2} = \frac{15}{2},$$

which cannot equal 6. Therefore the supplied data are inconsistent.

Conclusion.

No constants a, b, c satisfy all three conditions.

Verification. In fact, adding the two derivative equations gives $2(a + b + c) = 15$, forcing $q(1, 1) = 15/2$, not 6.

Question 9

Define

$$u(x, y) = \begin{cases} \frac{x^2 y}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Compute $u_x(0, 0)$ from the definition.
2. Compute $u_y(0, 0)$ from the definition.
3. Compute formulas for u_x, u_y away from the origin and explain why they do not determine the values at the origin.

Question 9 – Solution

Strategy. At the exceptional point, use one-variable difference quotients along the coordinate axes.

Step 1: x -partial at the origin

$$u_x(0,0) = \lim_{h \rightarrow 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \boxed{0}.$$

Step 2: y -partial at the origin

$$u_y(0,0) = \lim_{k \rightarrow 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k \rightarrow 0} \frac{0}{k} = \boxed{0}.$$

Step 3: Away from the origin Quotient differentiation gives

$$\boxed{u_x = \frac{2xy^3}{(x^2 + y^2)^2}}, \quad \boxed{u_y = \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2}} \quad ((x,y) \neq (0,0)).$$

These formulas were derived where the denominator is nonzero, so substituting $(0,0)$ would be invalid. The definition supplies the missing partial values.

Question 10

Define

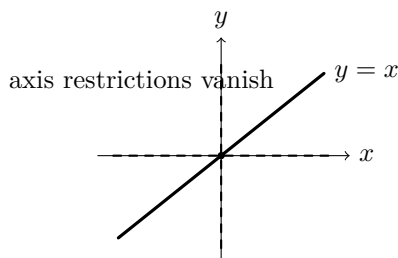
$$v(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Show that both first partial derivatives exist at the origin.
2. Test continuity along $y = x$.
3. Explain what this example proves about partial derivatives and continuity.

Question 10 – Solution

Strategy. Partial derivatives inspect axis restrictions only; compare those restrictions with a diagonal approach.



Step 1: Axis difference quotients Since $v(h, 0) = 0$ and $v(0, k) = 0$,

$$v_x(0, 0) = \lim_{h \rightarrow 0} \frac{v(h, 0) - v(0, 0)}{h} = 0,$$

$$v_y(0, 0) = \lim_{k \rightarrow 0} \frac{v(0, k) - v(0, 0)}{k} = 0.$$

Thus

$$\boxed{v_x(0, 0) = v_y(0, 0) = 0}.$$

Step 2: Diagonal path Along $y = x$ with $x \neq 0$,

$$v(x, x) = \frac{x^2}{2x^2} = \frac{1}{2}.$$

Therefore $\lim_{x \rightarrow 0} v(x, x) = 1/2 \neq v(0, 0) = 0$.

Conclusion. The function is not continuous at the origin even though both first partial derivatives exist there. Hence

$$\boxed{\text{existence of } v_x, v_y \text{ at a point does not imply continuity there.}}$$