

Question 1

Evaluate

$$\lim_{(x,y) \rightarrow (2,-1)} (x^2y + 3xy^2 - 2y + 5).$$

Tasks

1. State why direct substitution is valid.
2. Evaluate the limit exactly.
3. Verify the arithmetic term by term.

Question 1 – Solution

Strategy. A polynomial in two variables is continuous everywhere, so its limit equals its value at the target point.

Step 1: Continuity Each monomial and constant is continuous, and finite sums of continuous functions are continuous. Direct substitution is therefore valid.

Step 2: Substitute

$$\begin{aligned}x^2y + 3xy^2 - 2y + 5 &= (2)^2(-1) + 3(2)(-1)^2 - 2(-1) + 5 \\&= -4 + 6 + 2 + 5 = 9.\end{aligned}$$

Thus

$$\boxed{\lim_{(x,y) \rightarrow (2,-1)} (x^2y + 3xy^2 - 2y + 5) = 9}.$$

Verification. The four contributions are -4 , 6 , 2 , and 5 ; their sum is 9 . No denominator or restricted operation can fail at $(2, -1)$.

Question 2

Evaluate

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + xy - 3x - y + 2}{x - 1}.$$

Tasks

1. Explain why direct substitution is inconclusive.
2. Factor and simplify on the punctured domain.
3. Evaluate the limit and distinguish it from the function value.

Question 2 – Solution

Strategy. Factor the numerator to expose a common factor with the vanishing denominator.

Step 1: Indeterminate form At $(1, 2)$, numerator and denominator are both 0, so substitution gives $0/0$, not a limit value.

Step 2: Factor Treat the numerator as a polynomial in y :

$$x^2 - 3x + 2 + y(x - 1) = (x - 1)(x - 2) + y(x - 1) = (x - 1)(x + y - 2).$$

Whenever $x \neq 1$,

$$\frac{(x - 1)(x + y - 2)}{x - 1} = x + y - 2.$$

Step 3: Limit The simplified function is continuous, so

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x^2 + xy - 3x - y + 2}{x - 1} = 1.$$

The original quotient is undefined on the entire line $x = 1$, including the target, but points of its domain approach $(1, 2)$ and the missing line does not change the limit.

Question 3

Determine whether

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^4 + y^2}$$

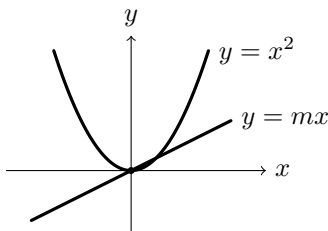
exists.

Tasks

1. Test every straight-line path $y = mx$.
2. Test a curved path suggested by the denominator.
3. Explain why the first test alone cannot prove existence.

Question 3 – Solution

Strategy. Compare straight lines with the balancing path $y = kx^2$, where x^4 and y^2 have the same order.



Step 1: Lines Along $y = mx$,

$$\frac{x^2(mx)}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \rightarrow 0$$

for $m \neq 0$; the path $y = 0$ also gives 0.

Step 2: Parabolas Along $y = kx^2$,

$$\frac{x^2(kx^2)}{x^4 + k^2x^4} = \frac{k}{1 + k^2}.$$

For example, $k = 1$ gives $1/2$, whereas $k = 0$ gives 0.

Conclusion. Since two paths produce different values,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2y}{x^4 + y^2} \text{ does not exist.}$$

Agreement along all lines is necessary but not sufficient because nonlinearly approaching paths remain possible.

Question 4

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2}.$$

Tasks

1. Establish a useful bound without selecting paths.
2. Apply the squeeze theorem.
3. Give an independent polar-coordinate check.

Question 4 – Solution

Strategy. Bound the numerator using $x^2 \leq x^2 + y^2$ and then compare with a quantity tending to zero.

Step 1: Bound For $(x, y) \neq (0, 0)$,

$$0 \leq \frac{x^2 y^2}{x^2 + y^2} \leq \frac{(x^2 + y^2) y^2}{x^2 + y^2} = y^2.$$

As $(x, y) \rightarrow (0, 0)$, $y^2 \rightarrow 0$.

Step 2: Squeeze Both bounding functions tend to 0, hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^2 + y^2} = 0}.$$

Step 3: Polar check With $x = r \cos \theta$ and $y = r \sin \theta$,

$$\frac{x^2 y^2}{x^2 + y^2} = r^2 \cos^2 \theta \sin^2 \theta.$$

Because $0 \leq \cos^2 \theta \sin^2 \theta \leq 1/4$, its absolute value is at most $r^2/4 \rightarrow 0$, uniformly in direction.

Question 5

For

$$f(x, y) = \frac{xy}{x^2 + y^2}, \quad (x, y) \neq (0, 0),$$

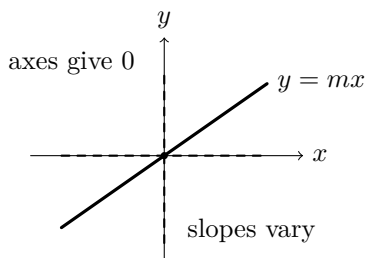
compare its two iterated limits with its joint limit at the origin.

Tasks

1. Compute both iterated limits.
2. Test the joint limit along $y = mx$.
3. Explain why the conclusions differ.

Question 5 – Solution

Strategy. Iterated limits approach along coordinate-aligned stages; the joint limit must agree along every path.



Step 1: Iterated limits For fixed $y \neq 0$, $\lim_{x \rightarrow 0} xy/(x^2 + y^2) = 0$, so

$$\lim_{y \rightarrow 0} \left(\lim_{x \rightarrow 0} f(x, y) \right) = 0.$$

Similarly, for fixed $x \neq 0$, the inner y -limit is 0, giving

$$\lim_{x \rightarrow 0} \left(\lim_{y \rightarrow 0} f(x, y) \right) = 0.$$

Step 2: Joint paths Along $y = mx$,

$$f(x, mx) = \frac{mx^2}{x^2 + m^2x^2} = \frac{m}{1 + m^2}.$$

This is 0 for $m = 0$ but $1/2$ for $m = 1$.

Conclusion. Both iterated limits equal $\boxed{0}$, yet the joint limit $\boxed{\text{does not exist}}$. Iterated limits inspect only two staged approaches and cannot control arbitrary simultaneous approaches.

Question 6

Let

$$D = \{(x, y) : x \geq 0, 0 \leq y \leq \sqrt{3}x\}, \quad f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}.$$

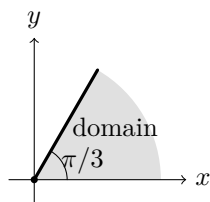
Find the limit of f as $(x, y) \rightarrow (0, 0)$ within D , or prove it does not exist.

Tasks

1. Describe the allowed directions.
2. Compare two valid boundary paths.
3. State the relative-limit conclusion.

Question 6 – Solution

Strategy. A relative limit uses only paths in the domain, but all such paths must still agree.



Step 1: Directions The wedge is $0 \leq \theta \leq \pi/3$. In polar form,

$$f = \frac{r^2 \cos^2 \theta - r^2 \sin^2 \theta}{r^2} = \cos 2\theta.$$

Step 2: Boundary paths Along $y = 0$ ($\theta = 0$), $f = 1$. Along $y = \sqrt{3}x$ ($\theta = \pi/3$),

$$f = \cos(2\pi/3) = -\frac{1}{2}.$$

Both paths stay in D and approach the origin.

Conclusion. Because the two permitted approaches disagree,

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ (x,y) \in D}} f(x,y) \text{ does not exist.}$$

Question 7

Evaluate

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right).$$

Tasks

1. Explain why the oscillatory factor has no limit.
2. Bound the full expression.
3. Prove the limit despite the oscillation.

Question 7 – Solution

Strategy. The sine factor is bounded even though it oscillates; the radial factor forces the product to zero.

Step 1: Absolute bound Since $|\sin u| \leq 1$,

$$\left| (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) \right| \leq x^2 + y^2.$$

Step 2: Squeeze As $(x, y) \rightarrow (0, 0)$, the right side tends to 0. Therefore

$$\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \sin\left(\frac{1}{x^2 + y^2}\right) = 0.$$

Step 3: Why oscillation is harmless Along radii for which $1/(x^2 + y^2)$ hits alternating sine peaks, the sine values do not settle. Nevertheless, the product's magnitude never exceeds the shrinking radius squared, so every path is controlled by the same bound.

Question 8

Prove directly from the ε - δ definition that

$$\lim_{(x,y) \rightarrow (1,-2)} (3x - 4y) = 11.$$

Tasks

1. Rewrite the output error.
2. Bound it by the Euclidean input distance.
3. Give an explicit valid choice of δ .

Question 8 – Solution

Strategy. Use Cauchy–Schwarz to compare the linear output change with the distance to $(1, -2)$.

Step 1: Error Let $h = x - 1$ and $k = y + 2$. Then

$$|(3x - 4y) - 11| = |3h - 4k|.$$

Step 2: Cauchy–Schwarz

$$|3h - 4k| \leq \sqrt{3^2 + (-4)^2} \sqrt{h^2 + k^2} = 5\sqrt{(x - 1)^2 + (y + 2)^2}.$$

Step 3: Choose δ Given $\varepsilon > 0$, set

$$\boxed{\delta = \varepsilon/5}.$$

If $0 < \sqrt{(x - 1)^2 + (y + 2)^2} < \delta$, then

$$|(3x - 4y) - 11| < 5\delta = \varepsilon.$$

This is exactly the two-variable limit definition, so the claimed limit is proved.

Question 9

Suppose (a, b) lies strictly inside the disk $x^2 + y^2 < 25$. Prove

$$\lim_{(x,y) \rightarrow (a,b)} \sqrt{25 - x^2 - y^2} = \sqrt{25 - a^2 - b^2},$$

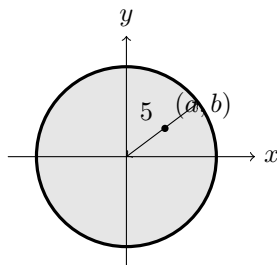
and give a concrete neighborhood that remains inside the domain.

Tasks

1. Find a positive safety margin to the boundary.
2. Choose a disk around (a, b) contained in the domain.
3. Justify the limit by composition.

Question 9 – Solution

Strategy. Use the triangle inequality to keep nearby points inside the radius-5 disk, then apply continuity.



Step 1: Margin Let $s = \sqrt{a^2 + b^2} < 5$. The radial margin is $5 - s > 0$.

Step 2: Neighborhood Choose

$$\delta_0 = \frac{5 - s}{2}.$$

If $\sqrt{(x - a)^2 + (y - b)^2} < \delta_0$, then by the triangle inequality

$$\sqrt{x^2 + y^2} \leq s + \sqrt{(x - a)^2 + (y - b)^2} < s + \frac{5 - s}{2} < 5.$$

Thus this entire neighborhood lies in the square-root domain.

Step 3: Composition The polynomial $25 - x^2 - y^2$ is continuous and remains positive there; $\sqrt{\cdot}$ is continuous on positive inputs. Their composition is continuous at (a, b) , proving the boxed limit stated in the question.

Question 10

Define

$$F(x, y) = \begin{cases} \frac{x^3 - 3xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ c, & (x, y) = (0, 0). \end{cases}$$

Tasks

1. Determine $\lim_{(x,y) \rightarrow (0,0)} F(x, y)$ using polar coordinates.
2. Find the unique value of c making F continuous at the origin.
3. Verify the estimate uniformly in angle.

Question 10 – Solution

Strategy. Recognize the cubic numerator's polar form and isolate one factor of radial distance.

Step 1: Convert With $x = r \cos \theta$ and $y = r \sin \theta$,

$$x^3 - 3xy^2 = r^3(\cos^3 \theta - 3 \cos \theta \sin^2 \theta) = r^3 \cos 3\theta.$$

Since $x^2 + y^2 = r^2$, for $r > 0$,

$$F(x, y) = r \cos 3\theta.$$

Step 2: Uniform estimate For every angle,

$$|F(x, y)| = r |\cos 3\theta| \leq r = \sqrt{x^2 + y^2} \longrightarrow 0.$$

Hence

$$\boxed{\lim_{(x,y) \rightarrow (0,0)} F(x, y) = 0}.$$

Step 3: Continuity value Continuity requires the assigned value to equal the limit, so the unique choice is

$$\boxed{c = 0}.$$

The bound is independent of θ , verifying that no directional or curved approach can change the result.