

Question 1

Find the exact length of

$$\mathbf{r}(t) = \left\langle 3t, 4t, \frac{1}{2}t^2 \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed and simplify it completely.
2. Set up and evaluate the arc-length integral.
3. Check that the result exceeds the straight-line distance between the endpoints.

Question 1 – Solution

Strategy. Arc length is the integral of speed: $L = \int_a^b \|\mathbf{r}'(t)\| dt$.

Step 1: Velocity and speed

$$\mathbf{r}'(t) = \langle 3, 4, t \rangle.$$

Therefore

$$\|\mathbf{r}'(t)\| = \sqrt{3^2 + 4^2 + t^2} = \sqrt{25 + t^2}.$$

Step 2: Integrate Use

$$\int \sqrt{t^2 + a^2} dt = \frac{1}{2} \left(t\sqrt{t^2 + a^2} + a^2 \ln \left| t + \sqrt{t^2 + a^2} \right| \right).$$

With $a = 5$,

$$\begin{aligned} L &= \int_0^2 \sqrt{25 + t^2} dt \\ &= \frac{1}{2} \left[t\sqrt{25 + t^2} + 25 \ln \left(t + \sqrt{25 + t^2} \right) \right]_0^2 \\ &= \boxed{\sqrt{29} + \frac{25}{2} \ln \left(\frac{2 + \sqrt{29}}{5} \right)}. \end{aligned}$$

Step 3: Endpoint check The displacement is

$$\mathbf{r}(2) - \mathbf{r}(0) = \langle 6, 8, 2 \rangle,$$

whose length is $\sqrt{36 + 64 + 4} = 2\sqrt{26} \approx 10.198$. The arc length is approximately 10.261, which is slightly larger, as a nonstraight path requires.

Question 2

A helix is parametrized by

$$\mathbf{r}(t) = \langle 5 \cos t, 5 \sin t, 12t \rangle, \quad 0 \leq t \leq 3\pi.$$

Tasks

1. Find its exact length.
2. Determine the length traveled during one complete revolution.
3. Explain why equal parameter intervals produce equal arc lengths here.

Question 2 – Solution

Strategy. Differentiate and look for constant speed.

Step 1: Speed

$$\mathbf{r}'(t) = \langle -5 \sin t, 5 \cos t, 12 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{25 \sin^2 t + 25 \cos^2 t + 144} \\ &= \sqrt{25 + 144} = 13.\end{aligned}$$

Step 2: Total length

$$L = \int_0^{3\pi} 13 \, dt = 13[t]_0^{3\pi} = \boxed{39\pi}.$$

Step 3: One revolution A complete revolution changes t by 2π . Therefore

$$L_{\text{rev}} = \int_t^{t+2\pi} 13 \, du = \boxed{26\pi}.$$

Because the speed is the same constant 13 at every parameter value, an interval of width Δt always contributes length $13\Delta t$. Thus equal parameter intervals have equal arc lengths.

Verification. The interval $[0, 3\pi]$ contains $3/2$ revolutions, and $(3/2)(26\pi) = 39\pi$, confirming the total.

Question 3

Determine the exact length of

$$\mathbf{r}(t) = \left\langle t - \frac{t^3}{3}, t^2, t + \frac{t^3}{3} \right\rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Compute the speed without prematurely expanding every square.
2. Identify and justify the perfect-square simplification.
3. Evaluate the length and verify the speed is never negative.

Question 3 – Solution

Strategy. Differentiate first and group symmetric components; the radicals collapse to a polynomial.

Step 1: Differentiate

$$\mathbf{r}'(t) = \langle 1 - t^2, 2t, 1 + t^2 \rangle.$$

Thus

$$\begin{aligned}\|\mathbf{r}'(t)\|^2 &= (1 - t^2)^2 + (2t)^2 + (1 + t^2)^2 \\ &= (1 - 2t^2 + t^4) + 4t^2 + (1 + 2t^2 + t^4) \\ &= 2 + 4t^2 + 2t^4 \\ &= 2(1 + t^2)^2.\end{aligned}$$

Step 2: Take the square root

$$\|\mathbf{r}'(t)\| = \sqrt{2}|1 + t^2|.$$

Since $1 + t^2 > 0$ for every real t , this becomes

$$\|\mathbf{r}'(t)\| = \sqrt{2}(1 + t^2).$$

Step 3: Integrate

$$\begin{aligned}L &= \sqrt{2} \int_0^2 (1 + t^2) dt \\ &= \sqrt{2} \left[t + \frac{t^3}{3} \right]_0^2 \\ &= \boxed{\frac{14\sqrt{2}}{3}}.\end{aligned}$$

The speed is strictly positive because $1 + t^2 > 0$, so the curve is regular throughout the interval.

Question 4

For

$$\mathbf{r}(t) = \langle 3t, 4t, 0 \rangle, \quad t \geq 1,$$

let $s(t)$ denote distance traveled from the starting parameter $t = 1$.

Tasks

1. Find $s(t)$ and its inverse $t(s)$.
2. Reparametrize the curve by arc length measured from $t = 1$.
3. Verify directly that the new parametrization has unit speed.

Question 4 – Solution

Strategy. Integrate speed from the specified base parameter, invert the resulting monotone function, and substitute.

Step 1: Arc-length function

$$\mathbf{r}'(t) = \langle 3, 4, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 5.$$

Therefore

$$s(t) = \int_1^t 5 \, du = 5(t - 1).$$

Since $t \geq 1$, we have $s \geq 0$, and solving for t gives

$$t(s) = 1 + \frac{s}{5}.$$

Step 2: Substitute

$$\begin{aligned} \mathbf{R}(s) &= \mathbf{r}\left(1 + \frac{s}{5}\right) \\ &= \left\langle 3 + \frac{3}{5}s, 4 + \frac{4}{5}s, 0 \right\rangle, \quad s \geq 0. \end{aligned}$$

Step 3: Unit-speed verification

$$\mathbf{R}'(s) = \left\langle \frac{3}{5}, \frac{4}{5}, 0 \right\rangle,$$

so

$$\|\mathbf{R}'(s)\| = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1.$$

Thus an increase of one unit in s corresponds to exactly one unit of distance along the curve.

Question 5

The upper semicircle $x^2 + y^2 = 4$ is represented by

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 0 \rangle, \quad 0 \leq t \leq \pi,$$

and also by

$$\mathbf{R}(u) = \langle 2 \cos(u^3), 2 \sin(u^3), 0 \rangle, \quad 0 \leq u \leq \sqrt[3]{\pi}.$$

Tasks

1. Compute the length using each parametrization.
2. Explain why the nonconstant speed of $\mathbf{R}$ does not change the result.
3. Identify the substitution connecting the two integrals.

Question 5 – Solution

Strategy. Calculate both speed integrals, taking care that $u \geq 0$ when simplifying an absolute value.

Step 1: Standard parametrization

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 0 \rangle, \quad \|\mathbf{r}'(t)\| = 2.$$

Hence

$$L_r = \int_0^\pi 2 \, dt = \boxed{2\pi}.$$

Step 2: Cubic reparametrization

$$\mathbf{R}'(u) = \langle -6u^2 \sin(u^3), 6u^2 \cos(u^3), 0 \rangle.$$

Because $u \geq 0$,

$$\|\mathbf{R}'(u)\| = 6u^2.$$

Therefore

$$L_R = \int_0^{\sqrt[3]{\pi}} 6u^2 \, du = [2u^3]_0^{\sqrt[3]{\pi}} = \boxed{2\pi}.$$

Step 3: Invariance The substitution $t = u^3$ gives $dt = 3u^2 du$, so

$$2 \, dt = 6u^2 \, du.$$

The second parametrization changes the rate at which points are visited, but it is increasing and covers the same semicircle once. The speed factor exactly compensates for the parameter distortion.

Question 6

A path travels from $A = (-2, 0, 0)$ to $C = (0, 2, 0)$ and then to $B = (2, 0, 0)$ according to

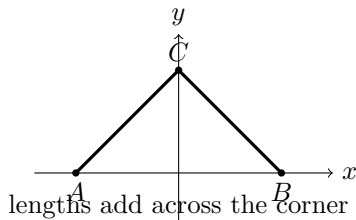
$$\mathbf{r}(t) = \begin{cases} \langle t, t+2, 0 \rangle, & -2 \leq t \leq 0, \\ \langle t, 2-t, 0 \rangle, & 0 < t \leq 2. \end{cases}$$

Tasks

1. Verify continuity at the joining parameter.
2. Compute the total arc length by splitting the integral.
3. Compare it with $\|B - A\|$ and explain the difference.

Question 6 – Solution

Strategy. Arc length is additive across a corner, even though the derivative changes abruptly there.



Step 1: Join check Both formulas give

$$\langle 0, 2, 0 \rangle$$

at $t = 0$ (the second by its right-hand limit), so the path is continuous at C .

Step 2: Length of each segment On the first branch,

$$\mathbf{r}'(t) = \langle 1, 1, 0 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{2}.$$

On the second branch,

$$\mathbf{r}'(t) = \langle 1, -1, 0 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{2}.$$

Therefore

$$\begin{aligned} L &= \int_{-2}^0 \sqrt{2} \, dt + \int_0^2 \sqrt{2} \, dt \\ &= 2\sqrt{2} + 2\sqrt{2} = \boxed{4\sqrt{2}}. \end{aligned}$$

Step 3: Chord comparison The endpoint distance is

$$\|B - A\| = \|\langle 4, 0, 0 \rangle\| = 4.$$

Since $4\sqrt{2} > 4$, the broken path is longer. Equality would require travel directly along the segment from A to B without the detour through C .

Question 7

For

$$\mathbf{r}(t) = \left\langle t^2, \frac{1}{6}(4t+1)^{3/2}, 0 \right\rangle, \quad t \geq 0,$$

find the parameter value $T > 0$ at which the distance traveled from $t = 0$ first equals $35/4$.

Tasks

1. Derive the arc-length function $s(T)$.
2. Solve the resulting equation exactly.
3. Explain why the solution is unique.

Question 7 – Solution

Strategy. Differentiate carefully; the squared speed is a perfect square on the stated domain.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, \sqrt{4t+1}, 0 \rangle.$$

For $t \geq 0$,

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 4t + 1} \\ &= \sqrt{(2t+1)^2} = |2t+1| = 2t+1.\end{aligned}$$

Step 2: Arc-length function

$$\begin{aligned}s(T) &= \int_0^T (2t+1) dt \\ &= [t^2 + t]_0^T = T^2 + T.\end{aligned}$$

Set this equal to the prescribed distance:

$$T^2 + T = \frac{35}{4}.$$

Multiplying by 4 and factoring,

$$4T^2 + 4T - 35 = (2T-5)(2T+7) = 0.$$

The candidates are $T = 5/2$ and $T = -7/2$; only the first lies in the domain. Hence

$$\boxed{T = \frac{5}{2}}.$$

Step 3: Uniqueness For $T > 0$, $s'(T) = 2T + 1 > 0$, so s is strictly increasing. Therefore the displayed equation has at most one positive solution. Since $s(0) = 0$ and $s(T) \rightarrow \infty$, it has exactly one. Direct substitution gives $s(5/2) = 25/4 + 10/4 = 35/4$, verifying the result.

Question 8

Find the exact length of the semicubical parabola

$$\mathbf{r}(t) = \langle t^2, t^3, 0 \rangle, \quad 0 \leq t \leq 2.$$

Tasks

1. Factor the speed using the sign of t .
2. Evaluate the arc-length integral by substitution.
3. Explain why the zero velocity at $t = 0$ does not make the length integral improper.

Question 8 – Solution

Strategy. Factor t^2 from the squared speed and use $t \geq 0$ before integrating.

Step 1: Speed

$$\mathbf{r}'(t) = \langle 2t, 3t^2, 0 \rangle.$$

Hence

$$\begin{aligned}\|\mathbf{r}'(t)\| &= \sqrt{4t^2 + 9t^4} \\ &= \sqrt{t^2(4 + 9t^2)} \\ &= |t|\sqrt{4 + 9t^2}.\end{aligned}$$

On $[0, 2]$, $|t| = t$, so the speed is $t\sqrt{4 + 9t^2}$.

Step 2: Integrate Let $u = 4 + 9t^2$, so $du = 18t \, dt$:

$$\begin{aligned}L &= \int_0^2 t\sqrt{4 + 9t^2} \, dt \\ &= \frac{1}{18} \int_4^{40} u^{1/2} \, du \\ &= \frac{1}{18} \left[\frac{2}{3} u^{3/2} \right]_4^{40} \\ &= \frac{1}{27} (40^{3/2} - 4^{3/2}) \\ &= \boxed{\frac{80\sqrt{10} - 8}{27}}.\end{aligned}$$

Step 3: Endpoint regularity Although $\mathbf{r}'(0) = \mathbf{0}$, the speed $t\sqrt{4 + 9t^2}$ is continuous at 0 and equals zero there. A continuous integrand on a closed interval defines an ordinary proper integral; no singularity is present.

Question 9

A wire is wound exactly N times around a cylinder of radius R , rising uniformly through total height H . Model its centerline by

$$\mathbf{r}(t) = \left\langle R \cos t, R \sin t, \frac{H}{2\pi N} t \right\rangle, \quad 0 \leq t \leq 2\pi N.$$

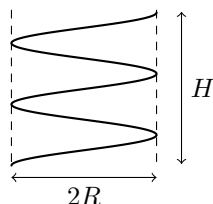
Tasks

1. Derive a formula for the wire length in terms of R, H, N .
2. For $R = 3$, $H = 24$, and length 30π , determine $N > 0$.
3. Verify whether the resulting value represents a whole number of turns.

Question 9 – Solution

Strategy. The model has constant speed; after integration, solve the design equation by squaring positive quantities.

wire wound around a cylinder



Step 1: General formula

$$\mathbf{r}'(t) = \left\langle -R \sin t, R \cos t, \frac{H}{2\pi N} \right\rangle,$$

so

$$\|\mathbf{r}'(t)\| = \sqrt{R^2 + \left(\frac{H}{2\pi N}\right)^2}.$$

Multiplying this constant speed by the parameter length $2\pi N$ gives

$$L = \sqrt{(2\pi NR)^2 + H^2}.$$

Step 2: Apply the data With $R = 3$, $H = 24$, and $L = 30\pi$,

$$(30\pi)^2 = (6\pi N)^2 + 24^2.$$

Therefore

$$\begin{aligned} 36\pi^2 N^2 &= 900\pi^2 - 576, \\ N^2 &= 25 - \frac{16}{\pi^2}. \end{aligned}$$

Since $N > 0$,

$$N = \sqrt{25 - \frac{16}{\pi^2}}.$$

Step 3: Feasibility Numerically, $N \approx 4.835$, which is not an integer. Thus the given continuous model has a positive solution, but the specifications are incompatible with a whole number of complete turns. This is a design-consistency conclusion, not an algebraic failure.

Question 10

Let $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^3$ be continuously differentiable.

Tasks

1. Prove the chord–arc inequality

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

2. Give a clear geometric interpretation.
3. Show that equality holds if $\mathbf{r}'(t) = \lambda(t)\mathbf{u}$, where $\mathbf{u}$ is fixed unit vector and $\lambda(t) \geq 0$.

Question 10 – Solution

Strategy. Express displacement as the integral of the velocity vector and apply a dot-product estimate in the displacement direction.

Step 1: Zero-displacement case If $\mathbf{r}(b) = \mathbf{r}(a)$, the left side is zero and the inequality follows because the integral of speed is nonnegative.

Now suppose $\mathbf{d} = \mathbf{r}(b) - \mathbf{r}(a) \neq \mathbf{0}$ and define the unit vector $\mathbf{u} = \mathbf{d}/\|\mathbf{d}\|$.

Step 2: Use the Fundamental Theorem

$$\mathbf{d} = \int_a^b \mathbf{r}'(t) dt.$$

Dot both sides with $\mathbf{u}$:

$$\begin{aligned}\|\mathbf{d}\| &= \mathbf{d} \cdot \mathbf{u} \\ &= \int_a^b \mathbf{r}'(t) \cdot \mathbf{u} dt \\ &\leq \int_a^b |\mathbf{r}'(t) \cdot \mathbf{u}| dt \\ &\leq \int_a^b \|\mathbf{r}'(t)\| \|\mathbf{u}\| dt \\ &= \int_a^b \|\mathbf{r}'(t)\| dt.\end{aligned}$$

Therefore

$$\|\mathbf{r}(b) - \mathbf{r}(a)\| \leq \int_a^b \|\mathbf{r}'(t)\| dt.$$

Step 3: Geometry and equality The straight chord is the shortest connection between the endpoints; following a curved or reversing path cannot be shorter. If $\mathbf{r}' = \lambda \mathbf{u}$ with $\lambda \geq 0$, then

$$\mathbf{r}(b) - \mathbf{r}(a) = \mathbf{u} \int_a^b \lambda(t) dt,$$

so its norm is $\int_a^b \lambda(t) dt$. Meanwhile $\|\mathbf{r}'\| = \lambda$, giving equality. The condition describes travel in one fixed direction without backtracking.