

Question 1

For

$$\mathbf{r}(t) = \langle t^2, 2t, t^2 - 2t \rangle,$$

find the unit tangent vector at $t = 1$ and write the tangent line there.

Tasks

1. Compute the point and velocity vector.
2. Normalize the velocity to obtain $\mathbf{T}(1)$.
3. Write and verify a tangent-line parametrization.

Question 1 – Solution

Strategy. At a regular point, the unit tangent is $\mathbf{T} = \mathbf{r}' / \|\mathbf{r}'\|$.

Step 1: Point and velocity

$$\mathbf{r}(1) = \langle 1, 2, -1 \rangle, \quad \mathbf{r}'(t) = \langle 2t, 2, 2t - 2 \rangle,$$

so $\mathbf{r}'(1) = \langle 2, 2, 0 \rangle \neq \mathbf{0}$.

Step 2: Normalize

$$\|\mathbf{r}'(1)\| = \sqrt{2^2 + 2^2} = 2\sqrt{2}.$$

Therefore

$$\mathbf{T}(1) = \frac{\langle 2, 2, 0 \rangle}{2\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

Step 3: Tangent line Either $\mathbf{r}'(1)$ or the parallel unit vector may be used:

$$\mathbf{L}(s) = \langle 1, 2, -1 \rangle + s \langle 1, 1, 0 \rangle.$$

At $s = 0$ the line passes through the curve point, and its direction $\langle 1, 1, 0 \rangle$ is one-half of $\mathbf{r}'(1)$.

Verification. The squared length of $\mathbf{T}(1)$ is $1/2 + 1/2 = 1$, as required.

Question 2

For the circular helix

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle,$$

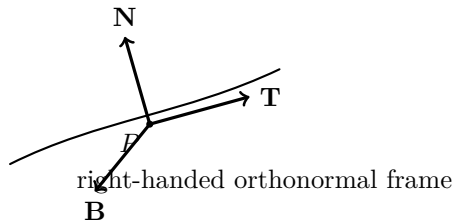
find the complete Frenet frame $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$.

Tasks

1. Normalize $\mathbf{r}'(t)$.
2. Normalize $\mathbf{T}'(t)$.
3. Compute $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ and verify orientation.

Question 2 – Solution

Strategy. The helix has constant speed, which makes each normalization especially transparent.

**Step 1: Unit tangent**

$$\mathbf{r}'(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{9 \sin^2 t + 9 \cos^2 t + 16} = 5.$$

Thus

$$\mathbf{T}(t) = \left\langle -\frac{3}{5} \sin t, \frac{3}{5} \cos t, \frac{4}{5} \right\rangle.$$

Step 2: Principal unit normal

$$\mathbf{T}'(t) = \left\langle -\frac{3}{5} \cos t, -\frac{3}{5} \sin t, 0 \right\rangle, \quad \|\mathbf{T}'(t)\| = \frac{3}{5}.$$

Therefore

$$\mathbf{N}(t) = \langle -\cos t, -\sin t, 0 \rangle.$$

Step 3: Binormal

$$\begin{aligned} \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \left\langle \frac{4}{5} \sin t, -\frac{4}{5} \cos t, \frac{3}{5} \right\rangle. \end{aligned}$$

Its norm is 1, and direct dot products give $\mathbf{T} \cdot \mathbf{N} = \mathbf{T} \cdot \mathbf{B} = \mathbf{N} \cdot \mathbf{B} = 0$. The defining cross product confirms the right-handed orientation.

Question 3

Consider the plane curve

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle.$$

At $t = 1$, determine $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$.

Tasks

1. Differentiate and normalize to find $\mathbf{T}(1)$.
2. Differentiate the general $\mathbf{T}(t)$ before evaluating to find $\mathbf{N}(1)$.
3. Compute $\mathbf{B}(1)$ and interpret its direction.

Question 3 – Solution

Strategy. Computing $\mathbf{T}'(1)$ requires differentiating the normalized tangent, not merely normalizing $\mathbf{r}''(1)$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 2t, 0 \rangle}{\sqrt{1 + 4t^2}}.$$

Hence

$$\boxed{\mathbf{T}(1) = \frac{1}{\sqrt{5}} \langle 1, 2, 0 \rangle}.$$

Step 2: Differentiate $\mathbf{T}$ Componentwise differentiation gives

$$\mathbf{T}'(t) = \frac{1}{(1 + 4t^2)^{3/2}} \langle -4t, 2, 0 \rangle.$$

At $t = 1$ this vector is a positive multiple of $\langle -2, 1, 0 \rangle$, whose length is $\sqrt{5}$. Therefore

$$\boxed{\mathbf{N}(1) = \frac{1}{\sqrt{5}} \langle -2, 1, 0 \rangle}.$$

Step 3: Binormal

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{5} \langle 1, 2, 0 \rangle \times \langle -2, 1, 0 \rangle = \boxed{\langle 0, 0, 1 \rangle}.$$

The curve lies in the xy -plane, so its binormal points perpendicular to that plane. The three vectors are unit length and mutually perpendicular.

Question 4

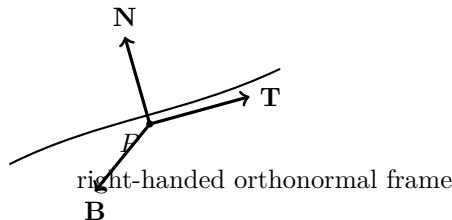
Let $\mathbf{r}$ be a regular curve for which $\mathbf{T}'(t) \neq \mathbf{0}$.

Tasks

1. Prove that $\mathbf{T}(t) \cdot \mathbf{N}(t) = 0$.
2. Prove that $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is a unit vector.
3. Explain why $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ is a right-handed orthonormal frame.

Question 4 – Solution

Strategy. Differentiate the invariant $\mathbf{T} \cdot \mathbf{T} = 1$, then use standard cross-product identities.



Step 1: Tangent-normal orthogonality Since $\mathbf{T}$ is a unit vector,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = 1.$$

Differentiate using the dot-product rule:

$$\mathbf{T}' \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{T}' = 0 \implies 2\mathbf{T} \cdot \mathbf{T}' = 0.$$

Because $\mathbf{N} = \mathbf{T}' / \|\mathbf{T}'\|$ and the denominator is nonzero,

$$\boxed{\mathbf{T} \cdot \mathbf{N} = 0}.$$

Step 2: Length of B For vectors separated by angle θ ,

$$\|\mathbf{T} \times \mathbf{N}\| = \|\mathbf{T}\| \|\mathbf{N}\| \sin \theta.$$

Here both lengths are 1 and $\theta = \pi/2$, so

$$\boxed{\|\mathbf{B}\| = 1}.$$

The cross product is perpendicular to both factors; hence $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

Step 3: Orientation The ordered relation $\mathbf{T} \times \mathbf{N} = \mathbf{B}$ is exactly the right-hand rule. Thus the three mutually perpendicular unit vectors form a right-handed orthonormal frame.

Question 5

The curve

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$$

is reparametrized by $t = 2u + 1$. Let $\mathbf{R}(u) = \mathbf{r}(2u + 1)$.

Tasks

1. Compare the unit tangents of $\mathbf{r}$ at $t = 2u + 1$ and $\mathbf{R}$ at u .
2. Compare their principal normals.
3. State what changes if the reparametrization is instead $t = 1 - 2u$.

Question 5 – Solution

Strategy. Apply the chain rule and keep track of the sign of the parameter derivative during normalization.

Step 1: Increasing reparametrization The chain rule gives

$$\mathbf{R}'(u) = 2\mathbf{r}'(2u + 1).$$

Since the factor 2 is positive,

$$\frac{\mathbf{R}'(u)}{\|\mathbf{R}'(u)\|} = \frac{2\mathbf{r}'(2u + 1)}{2\|\mathbf{r}'(2u + 1)\|} = \boxed{\mathbf{T}_r(2u + 1)}.$$

Thus an increasing reparametrization preserves the unit tangent.

Step 2: Principal normal Differentiating $\mathbf{T}_R(u) = \mathbf{T}_r(2u + 1)$ gives

$$\mathbf{T}'_R(u) = 2\mathbf{T}'_r(2u + 1).$$

Again the positive factor cancels upon normalization, so

$$\boxed{\mathbf{N}_R(u) = \mathbf{N}_r(2u + 1)}.$$

Step 3: Reversed parameter If $t = 1 - 2u$, then $dt/du = -2$. Consequently

$$\boxed{\mathbf{T}_R = -\mathbf{T}_r},$$

at corresponding points. Differentiating this relation with respect to u introduces a second negative factor, so $\mathbf{T}'_R = 2\mathbf{T}'_r$ and

$$\boxed{\mathbf{N}_R = \mathbf{N}_r}.$$

Therefore $\mathbf{B}_R = \mathbf{T}_R \times \mathbf{N}_R = -\mathbf{B}_r$. Reversing orientation changes $\mathbf{T}$ and $\mathbf{B}$, but not the principal normal.

Question 6

At a regular point of a curve, suppose

$$\mathbf{r}'(t_0) = \langle 1, 2, 2 \rangle, \quad \mathbf{r}''(t_0) = \langle 2, 0, 1 \rangle.$$

Tasks

1. Find the unit tangent $\mathbf{T}(t_0)$.
2. Find the binormal using $\mathbf{r}' \times \mathbf{r}''$.
3. Obtain the principal normal from $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ and verify the frame.

Question 6 – Solution

Strategy. The cross product $\mathbf{r}' \times \mathbf{r}''$ gives the binormal direction without differentiating a normalized vector.

Step 1: Tangent

$$\|\mathbf{r}'(t_0)\| = \sqrt{1+4+4} = 3, \quad \boxed{\mathbf{T}(t_0) = \frac{1}{3} \langle 1, 2, 2 \rangle}.$$

Step 2: Binormal direction

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 2 & 0 & 1 \end{vmatrix} = \langle 2, 3, -4 \rangle.$$

Its length is $\sqrt{4+9+16} = \sqrt{29}$, so

$$\boxed{\mathbf{B}(t_0) = \frac{1}{\sqrt{29}} \langle 2, 3, -4 \rangle}.$$

Step 3: Principal normal In a right-handed frame, $\mathbf{B} \times \mathbf{T} = \mathbf{N}$. Thus

$$\begin{aligned} \mathbf{N}(t_0) &= \frac{1}{3\sqrt{29}} \langle 2, 3, -4 \rangle \times \langle 1, 2, 2 \rangle \\ &= \boxed{\frac{1}{3\sqrt{29}} \langle 14, -8, 1 \rangle}. \end{aligned}$$

Its squared numerator length is $196 + 64 + 1 = 261 = 9(29)$, so it has unit length. Direct dot products with $\mathbf{T}$ and $\mathbf{B}$ are zero, verifying the frame.

Question 7

For

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

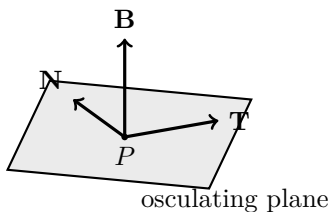
find equations of the osculating plane and normal plane at $t = 1$.

Tasks

1. Compute suitable directions for $\mathbf{T}$ and $\mathbf{B}$.
2. Use $\mathbf{B}$ as the osculating-plane normal.
3. Use $\mathbf{T}$ as the normal-plane normal and verify both planes pass through the curve point.

Question 7 – Solution

Strategy. The osculating plane is spanned by $\mathbf{T}$ and $\mathbf{N}$, so its normal is $\mathbf{B}$. The normal plane is perpendicular to $\mathbf{T}$.

**Step 1: Point and tangent direction**

$$P = \mathbf{r}(1) = (1, 1, 1), \quad \mathbf{r}'(1) = \langle 1, 2, 3 \rangle.$$

We may use $\langle 1, 2, 3 \rangle$ as the normal vector of the normal plane.

Step 2: Binormal direction

$$\mathbf{r}''(t) = \langle 0, 2, 6t \rangle, \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$

Then

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle = \langle 6, -6, 2 \rangle = 2 \langle 3, -3, 1 \rangle.$$

Hence an osculating-plane normal is $\langle 3, -3, 1 \rangle$.

Step 3: Plane equations Through $P = (1, 1, 1)$, the osculating plane is

$$3(x - 1) - 3(y - 1) + (z - 1) = 0 \implies \boxed{3x - 3y + z = 1}.$$

The normal plane is

$$(x - 1) + 2(y - 1) + 3(z - 1) = 0 \implies \boxed{x + 2y + 3z = 6}.$$

Substitution of $(1, 1, 1)$ verifies both equations, and their normals have the required Frenet directions.

Question 8

Consider

$$\mathbf{r}(t) = \langle t, t^3, 0 \rangle.$$

Tasks

1. Find $\mathbf{T}(0)$.
2. Determine whether the principal normal and binormal are defined at $t = 0$ by the standard Frenet formulas.
3. Compute the one-sided limiting normal directions and explain the geometric issue.

Question 8 – Solution

Strategy. Regularity of $\mathbf{r}$ guarantees $\mathbf{T}$, but defining $\mathbf{N}$ additionally requires $\mathbf{T}' \neq \mathbf{0}$.

Step 1: Unit tangent

$$\mathbf{r}'(t) = \langle 1, 3t^2, 0 \rangle, \quad \mathbf{T}(t) = \frac{\langle 1, 3t^2, 0 \rangle}{\sqrt{1 + 9t^4}}.$$

Thus $\boxed{\mathbf{T}(0) = \langle 1, 0, 0 \rangle}$.

Step 2: Differentiate $\mathbf{T}$ Differentiation and simplification give

$$\mathbf{T}'(t) = \frac{1}{(1 + 9t^4)^{3/2}} \langle -18t^3, 6t, 0 \rangle = \frac{6t}{(1 + 9t^4)^{3/2}} \langle -3t^2, 1, 0 \rangle.$$

Therefore $\mathbf{T}'(0) = \mathbf{0}$. The expression

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{\|\mathbf{T}'(0)\|}$$

would divide by zero, so $\boxed{\mathbf{N}(0) \text{ and } \mathbf{B}(0) \text{ are not defined by the Frenet formulas}}$.

Step 3: One-sided behavior For $t > 0$, normalization yields a vector tending to $\langle 0, 1, 0 \rangle$. For $t < 0$, the scalar factor $6t$ reverses direction, giving the limit $\langle 0, -1, 0 \rangle$. The unequal limits show that the preferred bending direction flips at the inflection point. Although the curve has a tangent there, it has no single principal-normal direction.

Question 9

Let

$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t \rangle.$$

Find all $t \in [0, 2\pi)$ for which the unit tangent is perpendicular to the fixed vector $\mathbf{a} = \langle 1, 1, 0 \rangle$, and find the corresponding points.

Tasks

1. Explain why it is enough to test $\mathbf{r}'(t) \cdot \mathbf{a} = 0$.
2. Solve the resulting trigonometric equation exactly.
3. Evaluate the curve at every admissible parameter.

Question 9 – Solution

Strategy. Normalizing a nonzero vector multiplies it by a positive scalar, which does not change whether its dot product with $\mathbf{a}$ is zero.

Step 1: Tangent direction

$$\mathbf{r}'(t) = \langle -2 \sin t, 2 \cos t, 1 \rangle, \quad \|\mathbf{r}'(t)\| = \sqrt{5}.$$

Thus $\mathbf{T} \cdot \mathbf{a} = 0$ exactly when $\mathbf{r}' \cdot \mathbf{a} = 0$.

Step 2: Solve

$$\begin{aligned} \mathbf{r}'(t) \cdot \mathbf{a} &= -2 \sin t + 2 \cos t = 0 \\ \implies \sin t &= \cos t \\ \implies \tan t &= 1. \end{aligned}$$

On $[0, 2\pi)$,

$$t = \frac{\pi}{4}, \frac{5\pi}{4}.$$

Neither solution is lost by using tangent because $\cos t \neq 0$ at both values.

Step 3: Points

$$\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle \sqrt{2}, \sqrt{2}, \frac{\pi}{4} \right\rangle,$$

and

$$\mathbf{r}\left(\frac{5\pi}{4}\right) = \left\langle -\sqrt{2}, -\sqrt{2}, \frac{5\pi}{4} \right\rangle.$$

At either parameter, substituting into $-2 \sin t + 2 \cos t$ gives zero, verifying perpendicularity.

Question 10

At a point $P = (1, -1, 2)$ on a regular curve, a unit tangent and a candidate principal normal are

$$\mathbf{T} = \frac{1}{3} \langle 2, 1, 2 \rangle, \quad \mathbf{N} = \frac{1}{3\sqrt{5}} \langle -4, 7, \frac{1}{2} \rangle.$$

Tasks

1. Determine whether the supplied vectors can belong to a Frenet frame, and diagnose any defect.
2. Replace $\mathbf{N}$ by the unit vector in the direction of the component of $\langle -4, 7, 1 \rangle$ perpendicular to $\mathbf{T}$.
3. Find the corrected $\mathbf{B}$ and the osculating plane at P .

Question 10 – Solution

Strategy. First audit unit length and orthogonality. Then use vector projection to remove the tangential component from the proposed bending direction.

Step 1: Diagnose the supplied normal The tangent is unit because $(4 + 1 + 4)/9 = 1$. But

$$\mathbf{T} \cdot \mathbf{N} = \frac{2(-4) + 1(7) + 2(1/2)}{9\sqrt{5}} = 0,$$

while

$$\|\mathbf{N}\|^2 = \frac{16 + 49 + 1/4}{45} = \frac{29}{20} \neq 1.$$

Thus it is perpendicular to $\mathbf{T}$ but is not a unit vector, so the data are not a Frenet frame.

Step 2: Correct the bending direction Let $\mathbf{w} = \langle -4, 7, 1 \rangle$. Its tangential scalar component is

$$\mathbf{w} \cdot \mathbf{T} = \frac{-8 + 7 + 2}{3} = \frac{1}{3}.$$

Therefore

$$\mathbf{w}_\perp = \mathbf{w} - (\mathbf{w} \cdot \mathbf{T})\mathbf{T} = \langle -4, 7, 1 \rangle - \frac{1}{9} \langle 2, 1, 2 \rangle = \frac{1}{9} \langle -38, 62, 7 \rangle.$$

Since $\sqrt{(-38)^2 + 62^2 + 7^2} = \sqrt{5337} = 3\sqrt{593}$,

$$\hat{\mathbf{N}} = \frac{1}{3\sqrt{593}} \langle -38, 62, 7 \rangle.$$

Step 3: Binormal and osculating plane

$$\mathbf{B} = \mathbf{T} \times \hat{\mathbf{N}} = \frac{1}{\sqrt{593}} \langle -13, -10, 18 \rangle.$$

The numerator has length $\sqrt{169 + 100 + 324} = \sqrt{593}$. Since $\mathbf{B}$ is normal to the osculating plane through P ,

$$-13(x - 1) - 10(y + 1) + 18(z - 2) = 0,$$

or

$$13x + 10y - 18z + 33 = 0.$$

Substitution of P gives zero, and the construction guarantees a right-handed orthonormal frame.