

Question 1

For $t \neq 0$, let

$$\mathbf{r}(t) = \left\langle \frac{\sin 3t}{t}, \frac{e^{2t} - 1}{t}, \frac{1 - \cos t}{t^2} \right\rangle.$$

Tasks

1. Compute $\lim_{t \rightarrow 0} \mathbf{r}(t)$.
2. Define $\mathbf{r}(0)$ so that $\mathbf{r}$ is continuous at 0.
3. Justify why a vector limit exists precisely when all component limits exist.

Question 1 – Solution

Strategy. A vector limit is evaluated component by component. Rewrite each component in terms of a standard scalar limit.

Step 1: First component

$$\frac{\sin 3t}{t} = 3 \frac{\sin 3t}{3t} \longrightarrow 3(1) = 3.$$

Step 2: Second component Put $u = 2t$. Then

$$\frac{e^{2t} - 1}{t} = 2 \frac{e^u - 1}{u} \longrightarrow 2.$$

Step 3: Third component Multiply by the conjugate:

$$\begin{aligned} \frac{1 - \cos t}{t^2} &= \frac{(1 - \cos t)(1 + \cos t)}{t^2(1 + \cos t)} \\ &= \frac{\sin^2 t}{t^2(1 + \cos t)} = \left(\frac{\sin t}{t} \right)^2 \frac{1}{1 + \cos t} \longrightarrow \frac{1}{2}. \end{aligned}$$

Therefore

$$\boxed{\lim_{t \rightarrow 0} \mathbf{r}(t) = \langle 3, 2, \tfrac{1}{2} \rangle}.$$

Step 4: Continuous extension Define $\boxed{\mathbf{r}(0) = \langle 3, 2, \tfrac{1}{2} \rangle}$. This makes the function value equal its limit. The vector limit criterion follows because the distance to a candidate vector tends to zero exactly when each coordinate difference tends to zero.

Question 2

Let

$$\mathbf{r}(t) = \langle t^2 - 1, t^3 - 3t, 2t + 1 \rangle.$$

Tasks

1. Find $\mathbf{r}'(t)$ directly from the vector difference quotient.
2. Find the tangent line at the point corresponding to $t = 1$.
3. Verify that your direction vector is nonzero.

Question 2 – Solution

Strategy. Expand $\mathbf{r}(t+h) - \mathbf{r}(t)$ componentwise, divide by h , and then let $h \rightarrow 0$.

Step 1: Difference quotient

$$\begin{aligned}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} &= \left\langle \frac{(t+h)^2 - t^2}{h}, \frac{(t+h)^3 - 3(t+h) - (t^3 - 3t)}{h}, \frac{2h}{h} \right\rangle \\ &= \langle 2t+h, 3t^2 + 3th + h^2 - 3, 2 \rangle.\end{aligned}$$

Taking $h \rightarrow 0$ yields

$$\boxed{\mathbf{r}'(t) = \langle 2t, 3t^2 - 3, 2 \rangle}.$$

Step 2: Point and direction At $t = 1$,

$$\mathbf{r}(1) = \langle 0, -2, 3 \rangle, \quad \mathbf{r}'(1) = \langle 2, 0, 2 \rangle.$$

The derivative is nonzero, so it supplies a valid tangent direction.

Step 3: Tangent line Using a new line parameter s ,

$$\boxed{\mathbf{L}(s) = \langle 0, -2, 3 \rangle + s \langle 2, 0, 2 \rangle}.$$

Substitution at $s = 0$ confirms the point, and the line direction is parallel to $\mathbf{r}'(1)$.

Question 3

Let

$$\mathbf{u}(t) = \langle t, t^2, 1 \rangle, \quad \mathbf{v}(t) = \langle \cos t, \sin t, t \rangle.$$

Without expanding first, compute at $t = 0$:

Tasks

1. $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}),$
2. $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}),$
3. $\frac{d}{dt}[(t^2 + 1)\mathbf{v}(t)].$

State the differentiation rule used in each part.

Question 3 – Solution

Strategy. Apply the dot-product, cross-product, and scalar–vector product rules before inserting $t = 0$.

At $t = 0$,

$$\mathbf{u} = \langle 0, 0, 1 \rangle, \quad \mathbf{u}' = \langle 1, 0, 0 \rangle, \quad \mathbf{v} = \langle 1, 0, 0 \rangle, \quad \mathbf{v}' = \langle 0, 1, 1 \rangle.$$

Step 1: Dot product rule

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) &= \mathbf{u}' \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v}', \\ \left. \frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) \right|_0 &= \langle 1, 0, 0 \rangle \cdot \langle 1, 0, 0 \rangle + \langle 0, 0, 1 \rangle \cdot \langle 0, 1, 1 \rangle = \boxed{2}. \end{aligned}$$

Step 2: Cross product rule Order matters:

$$\begin{aligned} \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) &= \mathbf{u}' \times \mathbf{v} + \mathbf{u} \times \mathbf{v}', \\ \left. \frac{d}{dt}(\mathbf{u} \times \mathbf{v}) \right|_0 &= \mathbf{0} + \langle 0, 0, 1 \rangle \times \langle 0, 1, 1 \rangle = \boxed{\langle -1, 0, 0 \rangle}. \end{aligned}$$

Step 3: Scalar–vector rule With $f(t) = t^2 + 1$,

$$(f\mathbf{v})' = f'\mathbf{v} + f\mathbf{v}' \implies (f\mathbf{v})'(0) = 0\mathbf{v}(0) + 1\mathbf{v}'(0) = \boxed{\langle 0, 1, 1 \rangle}.$$

Question 4

Find the unique vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = \left\langle 2te^{t^2}, \frac{3}{1+t^2}, 4t^3 - 2t \right\rangle, \quad \mathbf{r}(0) = \langle 1, -2, 5 \rangle.$$

Tasks

1. Recover $\mathbf{r}(t)$ by componentwise antidifferentiation.
2. Use the initial vector to determine every constant.
3. Differentiate your final expression to verify it.

Question 4 – Solution

Strategy. Antidifferentiate each component and then use the initial vector to determine three constants.

Step 1: First component

$$\int 2te^{t^2} dt = e^{t^2} + C_1,$$

using the substitution $u = t^2$.

Step 2: Remaining components

$$\int \frac{3}{1+t^2} dt = 3 \arctan t + C_2,$$

and

$$\int (4t^3 - 2t) dt = t^4 - t^2 + C_3.$$

Thus

$$\mathbf{r}(t) = \left\langle e^{t^2} + C_1, 3 \arctan t + C_2, t^4 - t^2 + C_3 \right\rangle.$$

Step 3: Initial condition At $t = 0$,

$$\mathbf{r}(0) = \langle 1 + C_1, C_2, C_3 \rangle = \langle 1, -2, 5 \rangle.$$

Hence $C_1 = 0$, $C_2 = -2$, and $C_3 = 5$. Therefore

$$\boxed{\mathbf{r}(t) = \left\langle e^{t^2}, 3 \arctan t - 2, t^4 - t^2 + 5 \right\rangle}.$$

Verification. Componentwise differentiation gives $\left\langle 2te^{t^2}, 3/(1+t^2), 4t^3 - 2t \right\rangle$, and direct substitution gives the required initial vector. These three scalar initial-value problems each have only one integration constant, so the solution is unique.

Question 5

Evaluate exactly

$$\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle dt.$$

Tasks

1. Evaluate all three components exactly, showing each scalar integration step.
2. State the resulting vector.
3. Verify the third component using a double-angle identity.

Question 5 – Solution

Strategy. Integrate componentwise. The first two components require integration by parts; the third can be checked in two different ways.

Step 1: First component Integration by parts with $u = t$, $dv = \cos t \, dt$ gives

$$\begin{aligned}\int_0^{\pi/2} t \cos t \, dt &= [t \sin t]_0^{\pi/2} - \int_0^{\pi/2} \sin t \, dt \\ &= \frac{\pi}{2} - 1.\end{aligned}$$

Step 2: Second component With $u = t$, $dv = \sin t \, dt$,

$$\int_0^{\pi/2} t \sin t \, dt = [-t \cos t]_0^{\pi/2} + \int_0^{\pi/2} \cos t \, dt = 1.$$

Step 3: Third component Since $d(\sin^2 t)/dt = 2 \sin t \cos t$,

$$\int_0^{\pi/2} \sin t \cos t \, dt = \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} = \frac{1}{2}.$$

Therefore

$$\boxed{\int_0^{\pi/2} \langle t \cos t, t \sin t, \sin t \cos t \rangle \, dt = \left\langle \frac{\pi}{2} - 1, 1, \frac{1}{2} \right\rangle}.$$

Using $\sin t \cos t = \frac{1}{2} \sin 2t$ gives the same third component: $[-\frac{1}{4} \cos 2t]_0^{\pi/2} = 1/2$.

Question 6

Suppose a differentiable vector function $\mathbf{r}(t)$ satisfies

$$\|\mathbf{r}(t)\| = 7$$

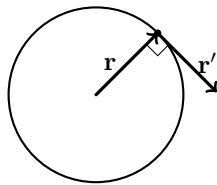
for every t in an interval.

Tasks

1. Prove that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ throughout the interval.
2. Explain the geometric meaning when $\mathbf{r}'(t) \neq \mathbf{0}$.
3. Show that $\mathbf{r} \cdot \mathbf{r}' = 0$ on an interval implies constant length; explain what extra datum fixes that length at 7.

Question 6 – Solution

Strategy. Square the norm to avoid differentiating a square root, then use the dot-product rule.



constant distance $\Rightarrow$ perpendicular change

Step 1: Differentiate the invariant The hypothesis gives

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = \|\mathbf{r}(t)\|^2 = 49.$$

Differentiating both sides,

$$\mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 0.$$

Because the dot product is commutative, the two terms are equal. Hence

$$2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \quad \implies \quad \boxed{\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0}.$$

Step 2: Geometry The position vector points radially from the origin. When $\mathbf{r}'(t) \neq \mathbf{0}$, it points along the tangent to the curve. Their zero dot product means the tangent is perpendicular to the radius, as expected for motion constrained to a sphere.

Step 3: Converse Now assume $\mathbf{r} \cdot \mathbf{r}' = 0$. Then

$$\frac{d}{dt} \|\mathbf{r}(t)\|^2 = \frac{d}{dt} (\mathbf{r} \cdot \mathbf{r}) = 2\mathbf{r} \cdot \mathbf{r}' = 0.$$

Therefore $\|\mathbf{r}(t)\|^2$ is constant on the interval, so its nonnegative square root $\|\mathbf{r}(t)\|$ is also constant. The constant is 7 if $\|\mathbf{r}(t_0)\| = 7$ at one point of the interval.

Question 7

Define

$$\mathbf{r}(t) = \begin{cases} \langle t, t^2, t^3 \rangle, & t \leq 0, \\ \langle t, at, t^2 \rangle, & t > 0, \end{cases}$$

where a is a real constant.

Tasks

1. Determine all a for which $\mathbf{r}$ is continuous at 0.
2. Determine all a for which $\mathbf{r}$ is differentiable at 0.
3. For the differentiable case, find $\mathbf{r}'(0)$.

Question 7 – Solution

Strategy. Continuity compares one-sided function values; differentiability compares one-sided vector difference quotients.

Step 1: Continuity The defined value from the first branch is

$$\mathbf{r}(0) = \langle 0, 0, 0 \rangle.$$

From the left, $\langle t, t^2, t^3 \rangle \rightarrow \langle 0, 0, 0 \rangle$. From the right,

$$\langle t, at, t^2 \rangle \rightarrow \langle 0, 0, 0 \rangle$$

for every fixed real a . Therefore $\mathbf{r}$ is continuous at 0 for every real value of a .

Step 2: Left derivative As h approaches zero through negative values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, h, h^2 \rangle \longrightarrow \langle 1, 0, 0 \rangle.$$

Step 3: Right derivative As h approaches zero through positive values,

$$\frac{\mathbf{r}(h) - \mathbf{r}(0)}{h} = \langle 1, a, h \rangle \longrightarrow \langle 1, a, 0 \rangle.$$

The one-sided derivatives agree exactly when $a = 0$. Hence

$$\boxed{a = 0}, \quad \boxed{\mathbf{r}'(0) = \langle 1, 0, 0 \rangle}.$$

This also shows explicitly why continuity alone does not guarantee differentiability.

Question 8

Let

$$\mathbf{F}(t) = \int_1^{t^2} \left\langle e^u, \frac{1}{u}, \cos(\pi u) \right\rangle du, \quad t > 0.$$

Tasks

1. Find $\mathbf{F}'(t)$ without first evaluating the integral.
2. Evaluate $\mathbf{F}'(1)$ and $\mathbf{F}(1)$.
3. Explain the roles of the Fundamental Theorem of Calculus and the chain rule.

Question 8 – Solution

Strategy. Differentiate the vector integral componentwise. The moving upper limit contributes its derivative by the chain rule.

Step 1: General rule If

$$\mathbf{F}(t) = \int_a^{g(t)} \mathbf{f}(u) du,$$

then the Fundamental Theorem gives the derivative with respect to the upper limit, while the chain rule multiplies by $g'(t)$:

$$\mathbf{F}'(t) = \mathbf{f}(g(t))g'(t).$$

Step 2: Apply the rule Here $g(t) = t^2$, so $g'(t) = 2t$. Therefore

$$\begin{aligned}\mathbf{F}'(t) &= 2t \left\langle e^{t^2}, \frac{1}{t^2}, \cos(\pi t^2) \right\rangle \\ &= \left\langle 2te^{t^2}, \frac{2}{t}, 2t \cos(\pi t^2) \right\rangle.\end{aligned}$$

The restriction $t > 0$ ensures the interval between 1 and t^2 never crosses the singularity $u = 0$.

Step 3: Values at $t = 1$

$$\mathbf{F}'(1) = \langle 2e, 2, -2 \rangle.$$

Also, an integral with identical limits is the zero vector:

$$\mathbf{F}(1) = \int_1^1 \mathbf{f}(u) du = \langle 0, 0, 0 \rangle.$$

These results distinguish accumulated value from its instantaneous rate of change.

Question 9

Let

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \quad t \geq 0,$$

and reparametrize it by $t = s^2 + 1$.

Tasks

1. Write the composite vector function $\mathbf{R}(s) = \mathbf{r}(s^2 + 1)$.
2. Compute $\mathbf{R}'(s)$ directly.
3. Compute it again using the vector chain rule and reconcile the fact that $\mathbf{R}'(0) = \mathbf{0}$ while $\mathbf{r}'(1) \neq \mathbf{0}$.

Question 9 – Solution

Strategy. Compare direct componentwise differentiation with $(\mathbf{r} \circ g)' = \mathbf{r}'(g)g'$.

Step 1: Composition

$$\mathbf{R}(s) = \langle s^2 + 1, (s^2 + 1)^2, (s^2 + 1)^3 \rangle.$$

Step 2: Direct differentiation

$$\begin{aligned}\mathbf{R}'(s) &= \langle 2s, 2(s^2 + 1)(2s), 3(s^2 + 1)^2(2s) \rangle \\ &= \langle 2s, 4s(s^2 + 1), 6s(s^2 + 1)^2 \rangle.\end{aligned}$$

Step 3: Chain-rule check Since

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \quad g(s) = s^2 + 1,$$

we obtain

$$\begin{aligned}\mathbf{R}'(s) &= \mathbf{r}'(g(s))g'(s) \\ &= \langle 1, 2(s^2 + 1), 3(s^2 + 1)^2 \rangle (2s),\end{aligned}$$

which is identical to the direct result.

At $s = 0$, $g'(0) = 0$, so $\mathbf{R}'(0) = \mathbf{0}$ even though

$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \neq \mathbf{0}.$$

There is no contradiction: the new parameter momentarily stops changing the old parameter. The geometric curve is regular at $t = 1$, but this particular parametrization is not regular at $s = 0$.

Question 10

Find every differentiable vector function $\mathbf{r}(t)$ satisfying

$$\mathbf{r}'(t) = 2t\mathbf{r}(t), \quad \mathbf{r}(0) = \langle a, b, c \rangle,$$

where a, b, c are constants. Then prove directly that any two solutions with the same initial vector must be identical.

Tasks

1. Solve the vector initial-value problem component by component.
2. Verify the resulting formula by differentiation and substitution.
3. Prove uniqueness using an integrating factor applied to the difference of two solutions.

Question 10 – Solution

Strategy. The vector equation is three identical scalar equations. Solve them componentwise, then prove uniqueness with an integrating factor.

Step 1: Split into components Write $\mathbf{r} = \langle x, y, z \rangle$. Then

$$x' = 2tx, \quad y' = 2ty, \quad z' = 2tz.$$

For a nonzero component w , separation gives

$$\frac{w'}{w} = 2t \implies \ln |w| = t^2 + C \implies w = K e^{t^2}.$$

The formula also includes the zero solution when $K = 0$.

Step 2: Apply the initial vector Since $e^0 = 1$, the constants are a, b, c . Thus

$$\boxed{\mathbf{r}(t) = e^{t^2} \langle a, b, c \rangle}.$$

Indeed,

$$\mathbf{r}'(t) = 2te^{t^2} \langle a, b, c \rangle = 2t\mathbf{r}(t),$$

and $\mathbf{r}(0) = \langle a, b, c \rangle$.

Step 3: Direct uniqueness proof Suppose $\mathbf{r}_1$ and $\mathbf{r}_2$ share the initial value, and let $\mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2$. Then

$$\mathbf{d}' = 2t\mathbf{d}, \quad \mathbf{d}(0) = \mathbf{0}.$$

Multiply by e^{-t^2} and use the product rule:

$$\frac{d}{dt}(e^{-t^2} \mathbf{d}(t)) = -2te^{-t^2} \mathbf{d} + e^{-t^2} \mathbf{d}' = \mathbf{0}.$$

Hence $e^{-t^2} \mathbf{d}$ is constant and equals its value $\mathbf{0}$ at $t = 0$. Therefore $\mathbf{d} = \mathbf{0}$, proving $\boxed{\mathbf{r}_1 = \mathbf{r}_2}$.