

Question 1

Let

$$\mathbf{r}(t) = \left\langle \sqrt{t+2}, \ln(5-t), \frac{1}{t^2-1} \right\rangle.$$

Tasks

1. Determine the exact domain of $\mathbf{r}$.
2. Evaluate $\mathbf{r}(2)$.
3. Explain what fails at every excluded endpoint or isolated value.

Question 1 – Solution

Strategy. A vector function is defined only where all three component functions are defined. We therefore intersect their domains.

Step 1: Square-root condition Real values require

$$t + 2 \geq 0 \implies t \geq -2.$$

Step 2: Logarithm condition The logarithm requires a strictly positive argument:

$$5 - t > 0 \implies t < 5.$$

Step 3: Denominator condition We must have

$$t^2 - 1 \neq 0 \implies t \neq -1, 1.$$

Intersecting the three conditions gives

$$\boxed{\text{Dom}(\mathbf{r}) = [-2, -1) \cup (-1, 1) \cup (1, 5]}.$$

Step 4: Evaluation

$$\boxed{\mathbf{r}(2) = \left\langle 2, \ln 3, \frac{1}{3} \right\rangle}.$$

Verification. At $t = -2$ the square root is zero, so that endpoint is included. At $t = 5$ the logarithm sees zero; below -2 its first component is nonreal; and at $t = \pm 1$ the third component divides by zero.

Question 2

Consider

$$\mathbf{r}(t) = \langle 2t - 1, t^2 + 1, 3t + 4 \rangle, \quad -2 \leq t \leq 2.$$

Tasks

1. Eliminate t and give two Cartesian equations satisfied by the trace.
2. Identify the curve and its containing plane.
3. Find its two endpoints and indicate the direction of travel.

Question 2 – Solution

Strategy. Solve the simplest component for the parameter and substitute it into the other two components.

Step 1: Recover the parameter Since $x = 2t - 1$,

$$t = \frac{x+1}{2}.$$

Step 2: Eliminate t Substitution into $y = t^2 + 1$ gives

$$y = \frac{(x+1)^2}{4} + 1.$$

Also,

$$z = 3\left(\frac{x+1}{2}\right) + 4 = \frac{3}{2}x + \frac{11}{2},$$

or $3x - 2z + 11 = 0$. Hence the trace is the part of the parabola

$$\boxed{y = \frac{(x+1)^2}{4} + 1, \quad 3x - 2z + 11 = 0}$$

for which $-5 \leq x \leq 3$. It lies in the stated plane.

Step 3: Endpoints and orientation

$$\mathbf{r}(-2) = \langle -5, 5, -2 \rangle, \quad \mathbf{r}(2) = \langle 3, 5, 10 \rangle.$$

As t increases, $x = 2t - 1$ increases, so the curve is traversed from the first point to the second.

Verification. Substituting either endpoint into both Cartesian equations confirms both constraints exactly.

Question 3

A particle follows

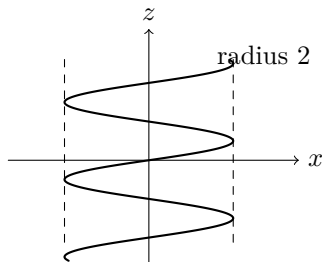
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, t/\pi \rangle, \quad 0 \leq t \leq 4\pi.$$

Without calculus:

1. Describe the trace geometrically and state its axis and radius.
2. Find every point where the particle meets the plane $z = 1$.
3. Determine the vertical rise during one complete revolution and the total number of revolutions.

Question 3 – Solution

Strategy. Use the identity $\cos^2 t + \sin^2 t = 1$ and compare the angular parameter with the height.



Step 1: Cylindrical constraint

$$x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4.$$

Thus the trace lies on the cylinder of radius 2 about the z -axis. Since $z = t/\pi$ steadily increases while (x, y) rotates counterclockwise, the curve is a circular helix.

Step 2: Plane intersection Setting $z = 1$ gives

$$\frac{t}{\pi} = 1 \implies t = \pi.$$

This value is in the interval, and therefore the only point is

$$\mathbf{r}(\pi) = \langle -2, 0, 1 \rangle.$$

Step 3: Pitch and turns One revolution changes t by 2π , so

$$\Delta z = \frac{t + 2\pi}{\pi} - \frac{t}{\pi} = 2.$$

The interval length 4π contains $4\pi/(2\pi) = 2$ revolutions. Hence the rise per turn is $\boxed{2}$ and the curve makes $\boxed{2}$ complete turns.

Question 4

Two moving objects have position functions

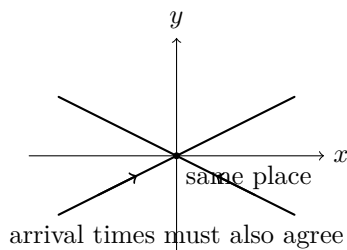
$$\mathbf{r}_1(t) = \langle t, t^2, 1 - t \rangle, \quad \mathbf{r}_2(s) = \langle 2 - s, s, s - 1 \rangle.$$

Tasks

1. Determine whether their geometric paths intersect.
2. If they do, find the parameter value on each path and the intersection point.
3. If both parameters represent the same clock time, decide whether the objects collide.

Question 4 – Solution

Strategy. A path intersection permits different parameters t and s . A collision additionally requires the same time.



Step 1: Match coordinates Equality of the two position vectors requires

$$t = 2 - s, \quad t^2 = s, \quad 1 - t = s - 1.$$

The first and third equations both simplify to

$$t + s = 2.$$

Using $s = t^2$ gives

$$t + t^2 = 2 \implies t^2 + t - 2 = 0 \implies (t - 1)(t + 2) = 0.$$

Step 2: Check both candidates If $t = 1$, then $s = t^2 = 1$, and all equations hold. If $t = -2$, then $s = 4$, but $t + s = 2$ also holds. Thus there are two path intersections:

$$\boxed{\langle 1, 1, 0 \rangle \quad (t, s) = (1, 1)}, \quad \boxed{\langle -2, 4, 3 \rangle \quad (t, s) = (-2, 4)}.$$

Step 3: Collision test With a common clock, a collision requires $t = s$. Only $(t, s) = (1, 1)$ satisfies that requirement. Therefore the objects collide at

$$\boxed{t = 1 \text{ at } \langle 1, 1, 0 \rangle}.$$

Verification. Direct substitution into both vector functions reproduces each path-intersection point.

Question 5

Construct a vector parametrization of the curve where

$$x^2 + y^2 = 9 \quad \text{and} \quad z = x - 2y.$$

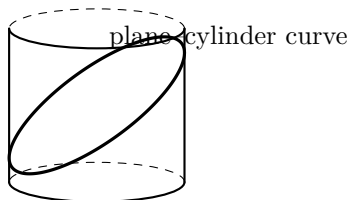
Choose the parametrization so that it begins at $(3, 0, 3)$ and initially moves toward points with $y > 0$. State a parameter interval that traces the curve exactly once.

Tasks

1. Parametrize the cylindrical constraint with the required starting direction.
2. Impose the plane equation to obtain the height component.
3. State and verify an interval that traces the curve exactly once.

Question 5 – Solution

Strategy. Parametrize the cylinder first; then use the plane equation to determine the remaining component.



Step 1: Parametrize the circle The cylinder equation suggests

$$x = 3 \cos t, \quad y = 3 \sin t,$$

because these expressions satisfy $x^2 + y^2 = 9$ identically.

Step 2: Impose the plane Substitute the chosen x and y into $z = x - 2y$:

$$z = 3 \cos t - 6 \sin t.$$

Therefore

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 3 \cos t - 6 \sin t \rangle, \quad 0 \leq t < 2\pi.$$

Step 3: Initial conditions At $t = 0$,

$$\mathbf{r}(0) = \langle 3, 0, 3 \rangle.$$

For small positive t , $\sin t > 0$, so $y = 3 \sin t > 0$; the required initial direction is satisfied.

Verification. Squaring the first two components gives $9(\cos^2 t + \sin^2 t) = 9$, and the third component is exactly the first minus twice the second. The half-open interval avoids repeating the starting point.

Question 6

The vector functions

$$\mathbf{r}(t) = \langle \cos t, \sin t, \cos 2t \rangle, \quad 0 \leq t < 2\pi,$$

and

$$\mathbf{q}(u) = \langle \cos u, -\sin u, \cos 2u \rangle, \quad 0 \leq u < 2\pi,$$

are given.

1. Prove that they have the same geometric trace.
2. Compare their orientations starting at $(1, 0, 1)$.
3. Decide whether either parametrization visits a point twice on its stated interval.

Question 6 – Solution

Strategy. Find a parameter substitution connecting the two formulas, then use the first two components to test repeated points.

Step 1: Same trace For $0 < u < 2\pi$, put $t = 2\pi - u$; at $u = 0$, use $t = 0$. For the interior parameters,

$$\cos(2\pi - u) = \cos u, \quad \sin(2\pi - u) = -\sin u,$$

and

$$\cos(2t) = \cos(4\pi - 2u) = \cos 2u.$$

Hence $\mathbf{r}(2\pi - u) = \mathbf{q}(u)$, so every point of one trace occurs on the other. Thus their geometric traces are identical.

Step 2: Orientation Both begin at $(1, 0, 1)$. For small positive t , the second component of $\mathbf{r}(t)$ is positive. For small positive u , the second component of $\mathbf{q}(u)$ is negative. Therefore they traverse the trace in opposite orientations.

Step 3: Repeated-point test Suppose $\mathbf{r}(t_1) = \mathbf{r}(t_2)$. Its first two coordinates give

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2),$$

so $t_1 - t_2$ is an integer multiple of 2π . Since both parameters lie in $[0, 2\pi)$, this forces $t_1 = t_2$. The same argument applies to $\mathbf{q}$.

Thus

each parametrization traces the curve exactly once, in opposite directions.

Question 7

Let

$$\mathbf{a}(t) = \langle t, 1 - t, 2t \rangle, \quad \mathbf{b}(t) = \langle 1, t, t^2 \rangle.$$

Tasks

1. Find the common domain of $\mathbf{a}/(t - 2)$ and $\mathbf{b}/(t + 1)$.
2. Compute $2\mathbf{a}(t) - \mathbf{b}(t)$.
3. Find all t for which $\mathbf{a}(t) \cdot \mathbf{b}(t) = 0$.

Question 7 – Solution

Strategy. Treat scalar division and vector algebra componentwise; for orthogonality, set the dot product equal to zero.

Step 1: Common domain The vector functions themselves are polynomial and defined for all real t . The scalar denominators impose

$$t - 2 \neq 0, \quad t + 1 \neq 0.$$

Therefore the common domain is

$$\boxed{\mathbb{R} \setminus \{-1, 2\}}.$$

Step 2: Vector combination

$$\begin{aligned} 2\mathbf{a}(t) - \mathbf{b}(t) &= \langle 2t, 2 - 2t, 4t \rangle - \langle 1, t, t^2 \rangle \\ &= \boxed{\langle 2t - 1, 2 - 3t, 4t - t^2 \rangle}. \end{aligned}$$

Step 3: Orthogonality

$$\begin{aligned} \mathbf{a}(t) \cdot \mathbf{b}(t) &= t(1) + (1 - t)t + (2t)t^2 \\ &= 2t - t^2 + 2t^3 \\ &= t(2t^2 - t + 2). \end{aligned}$$

The quadratic has discriminant $(-1)^2 - 4(2)(2) = -15 < 0$, so it has no real roots. Consequently

$$\boxed{t = 0}$$

is the only real solution. Directly, $\mathbf{a}(0) = \langle 0, 1, 0 \rangle$ and $\mathbf{b}(0) = \langle 1, 0, 0 \rangle$, whose dot product is zero.

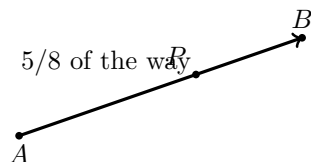
Question 8

Points $A = (-2, 1, 4)$ and $B = (6, -3, 0)$ are endpoints of a directed segment.

1. Construct a vector function $\mathbf{r}(t)$, $0 \leq t \leq 1$, that travels from A to B at a constant parameter rate.
2. Find the point reached when five-eighths of the parameter interval has elapsed.
3. Reparametrize the same directed segment on $-3 \leq u \leq 5$.

Question 8 – Solution

Strategy. A directed segment is an affine blend $(1 - t)A + tB$. A new interval requires a linear map back to $[0, 1]$.



Step 1: Segment formula First compute

$$B - A = \langle 8, -4, -4 \rangle.$$

Thus

$$\boxed{\mathbf{r}(t) = A + t(B - A) = \langle -2 + 8t, 1 - 4t, 4 - 4t \rangle}, \quad 0 \leq t \leq 1.$$

Indeed, $\mathbf{r}(0) = A$ and $\mathbf{r}(1) = B$.

Step 2: Division point At $t = 5/8$,

$$\mathbf{r}\left(\frac{5}{8}\right) = \left\langle -2 + 5, 1 - \frac{5}{2}, 4 - \frac{5}{2} \right\rangle = \boxed{\left\langle 3, -\frac{3}{2}, \frac{3}{2} \right\rangle}.$$

Step 3: New parameter interval The linear map taking $u = -3$ to $t = 0$ and $u = 5$ to $t = 1$ is

$$t = \frac{u + 3}{8}.$$

Substitution gives

$$\boxed{\mathbf{R}(u) = \left\langle u + 1, -\frac{1}{2}u - \frac{1}{2}, \frac{5}{2} - \frac{1}{2}u \right\rangle}, \quad -3 \leq u \leq 5.$$

Checking $u = -3$ and $u = 5$ recovers A and B in the required order.

Question 9

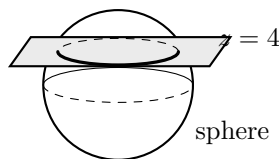
A curve is known to lie on the sphere $x^2 + y^2 + z^2 = 25$ and in the plane $z = 4$. Construct two vector parametrizations that traverse the entire curve exactly once in opposite directions. Choose both to start at $(3, 0, 4)$.

Tasks

1. Identify the intersection curve and its dimensions.
2. Construct the two oppositely oriented parametrizations.
3. Verify the surface constraints, starting point, and one-pass intervals.

Question 9 – Solution

Strategy. Substituting the fixed height into the sphere reveals a circle. The sign of the sine component controls orientation.



Step 1: Identify the circle With $z = 4$, the sphere equation becomes

$$x^2 + y^2 + 16 = 25 \implies x^2 + y^2 = 9.$$

The intersection is a radius-3 circle centered at $(0, 0, 4)$.

Step 2: First orientation A standard counterclockwise parametrization as viewed from the positive z -axis is

$$\mathbf{r}_+(t) = \langle 3 \cos t, 3 \sin t, 4 \rangle, \quad 0 \leq t < 2\pi.$$

It begins at $\mathbf{r}_+(0) = \langle 3, 0, 4 \rangle$ and initially enters the half-space $y > 0$.

Step 3: Opposite orientation Reverse the sign of the sine component:

$$\mathbf{r}_-(t) = \langle 3 \cos t, -3 \sin t, 4 \rangle, \quad 0 \leq t < 2\pi.$$

It has the same starting point but initially enters $y < 0$, so its orientation is opposite.

Verification. For either formula,

$$x^2 + y^2 + z^2 = 9(\cos^2 t + \sin^2 t) + 16 = 25.$$

The half-open interval covers every angular position once without duplicating the start at 2π .

Question 10

Consider the closed space curve

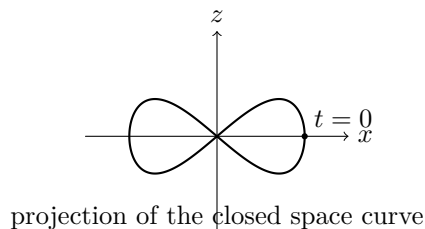
$$\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, \sin t \cos t \rangle, \quad 0 \leq t < 2\pi.$$

Tasks

1. Find two surfaces whose intersection contains the entire trace.
2. Determine all points where the trace meets the plane $z = 0$.
3. Prove that the parametrization traces each point exactly once, despite the repeated values of the z -component.

Question 10 – Solution

Strategy. Eliminate the parameter using the first two components. For uniqueness, retain the ordered pair $(\cos t, \sin t)$ rather than examining z alone.



Step 1: Eliminate the parameter The first two components give

$$x^2 + y^2 = 4(\cos^2 t + \sin^2 t) = 4.$$

Also,

$$xy = (2 \cos t)(2 \sin t) = 4 \sin t \cos t = 4z,$$

so $z = xy/4$. The trace is therefore contained in

$$\boxed{x^2 + y^2 = 4, \quad z = \frac{xy}{4}}.$$

Conversely, every point of this intersection has $(x/2, y/2)$ on the unit circle and hence equals $(\cos t, \sin t)$ for some t ; thus these equations describe the full trace.

Step 2: Intersections with $z = 0$ Since $z = xy/4$, we need $x = 0$ or $y = 0$. On $x^2 + y^2 = 4$, this yields

$$\boxed{(2, 0, 0), (-2, 0, 0), (0, 2, 0), (0, -2, 0)}.$$

Step 3: One-to-one traversal If $\mathbf{r}(t_1) = \mathbf{r}(t_2)$, then the first two coordinates imply

$$(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2).$$

Thus $t_1 - t_2 = 2\pi k$ for an integer k . Inside $[0, 2\pi)$, only $k = 0$ is possible, so $t_1 = t_2$. Hence no point is repeated.