

Question 1

Let

$$f(x, y) = \frac{\sqrt{9 - x^2 - y^2}}{\ln(x - y)}.$$

Tasks

1. Determine the exact domain.
2. State which boundary curves are included or excluded.
3. Test $(2, 0)$, $(0, 0)$, and $(1, 0)$ for membership.

Question 1 – Solution

Strategy Enforce separately the square-root, logarithm, and nonzero-denominator conditions.

Step 1: Square root Real values require $9 - x^2 - y^2 \geq 0$, hence $x^2 + y^2 \leq 9$.

Step 2: Logarithm The logarithm requires $x - y > 0$. Because it is a denominator, $\ln(x - y) \neq 0$, so $x - y \neq 1$.

Step 3: Combine Therefore

$$D = \{(x, y) : x^2 + y^2 \leq 9, x > y, x - y \neq 1\}.$$

The circular boundary is included only where the other restrictions hold; $x = y$ and $x - y = 1$ are excluded.

Step 4: Test points $(2, 0)$ satisfies all conditions. $(0, 0)$ fails $x > y$. $(1, 0)$ makes the denominator $\ln 1 = 0$. Thus only $\boxed{(2, 0)}$ belongs to D .

Question 2

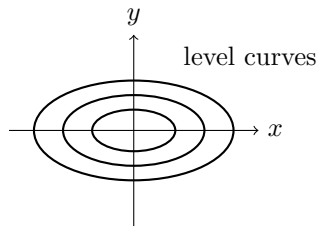
For $f(x, y) = x^2 + 4y^2$, analyze the level curves $f(x, y) = c$.

Tasks

1. Classify them for $c < 0$, $c = 0$, and $c > 0$.
2. Find semiaxes and enclosed area for $c > 0$.
3. Explain how the contours determine the graph's shape.

Question 2 – Solution

Strategy Treat c by sign and normalize the positive-level equation.



Step 1: Cases Because squares are nonnegative, $c < 0$ gives no points. For $c = 0$, both squares vanish, giving $(0, 0)$. For $c > 0$,

$$\frac{x^2}{c} + \frac{y^2}{c/4} = 1.$$

Thus the semiaxes are $\sqrt{c}$ and $\sqrt{c}/2$.

Step 2: Area The enclosed area is

$$A(c) = \pi (\sqrt{c}) \left(\frac{\sqrt{c}}{2} \right) = \boxed{\frac{\pi c}{2}}.$$

Step 3: Graph Higher levels produce nested, expanding ellipses centered at the origin. Since $z = x^2 + 4y^2 \geq 0$ and increases quadratically, the graph is an upward elliptic paraboloid, steeper in the y -direction.

Question 3

For positive inputs define

$$F(x, y) = \frac{xy}{x + y}.$$

Tasks

1. Show how F changes when both inputs are scaled by $s > 0$.
2. Derive the level curve $F = c > 0$.
3. State its domain branches and asymptotes.

Question 3 – Solution

Strategy Substitute scaled inputs, then solve the level equation explicitly for y .

Step 1: Scaling Direct substitution gives

$$F(sx, sy) = \frac{s^2 xy}{s(x+y)} = sF(x, y).$$

Thus the output has degree-one scaling.

Step 2: Level curve Set $xy/(x+y) = c$:

$$xy = cx + cy \implies y(x - c) = cx \implies \boxed{y = \frac{cx}{x - c}}.$$

For positive x, y , the denominator must be positive, so $x > c$; symmetry also implies $y > c$.

Step 3: Geometry Rewrite $y = c + c^2/(x - c)$. The asymptotes are $\boxed{x = c}$ and $\boxed{y = c}$, and the positive-input level is the branch with $x > c, y > c$. Substitution back into F returns c .

Question 4

A temperature field is

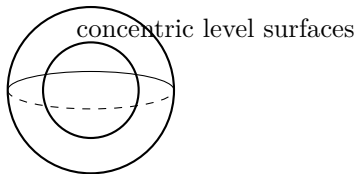
$$T(x, y, z) = 100e^{-(x^2+y^2+z^2)/4}.$$

Tasks

1. Describe the level surface $T = T_0$ for $0 < T_0 \leq 100$.
2. Find the 50-degree level radius.
3. Along $(t, t, 0)$, find when the temperature first reaches 50.

Question 4 – Solution

Strategy Take natural logarithms after isolating the exponential.



Step 1: General level From $T_0/100 = e^{-r^2/4}$,

$$\ln(T_0/100) = -r^2/4 \implies \boxed{r^2 = 4 \ln(100/T_0)}.$$

Thus levels are concentric spheres; $T_0 = 100$ degenerates to the origin.

Step 2: Fifty-degree surface For $T_0 = 50$, $r^2 = 4 \ln 2$, so $\boxed{r = 2\sqrt{\ln 2}}$.

Step 3: Path On $(t, t, 0)$, $r^2 = 2t^2$. Hence

$$2t^2 = 4 \ln 2 \implies t = \pm \sqrt{2 \ln 2}.$$

Starting at $t = 0$ and moving with increasing t , the first time is $\boxed{t = \sqrt{2 \ln 2}}$.

Question 5

Let $f(x, y) = \sqrt{1 - x^2 - y^2}$ and define $g(u, v) = f(u + v, u - v)$.

Tasks

1. Find a simplified formula and exact domain for g .
2. Describe its level curves.
3. Explain the geometric effect of the coordinate change.

Question 5 – Solution

Strategy Substitute first and expand carefully.

Step 1: Simplify Since

$$(u + v)^2 + (u - v)^2 = 2u^2 + 2v^2,$$

we get $\boxed{g(u, v) = \sqrt{1 - 2u^2 - 2v^2}}$.

Step 2: Domain The radicand must be nonnegative:

$$1 - 2u^2 - 2v^2 \geq 0 \implies \boxed{u^2 + v^2 \leq \frac{1}{2}}.$$

Step 3: Levels For $0 \leq c \leq 1$, $g = c$ implies

$$u^2 + v^2 = \frac{1 - c^2}{2},$$

circles centered at the origin. The map $(u, v) \mapsto (x, y) = (u + v, u - v)$ reflects v to $-v$, then rotates by 45° and scales lengths by $\sqrt{2}$. Its inverse image of the unit disk is the disk of radius $1/\sqrt{2}$. At $c = 1$ the level is just the origin; levels outside $[0, 1]$ are empty.

Question 6

Determine the exact range of

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad (x, y) \neq (0, 0).$$

Tasks

1. Prove upper and lower bounds.
2. Show every intermediate value occurs.
3. Describe the level sets $f = c$.

Question 6 – Solution

Strategy Compare $|x^2 - y^2|$ with $x^2 + y^2$, then solve the level equation.

Step 1: Bounds Since $x^2, y^2 \geq 0$,

$$|x^2 - y^2| \leq x^2 + y^2.$$

The denominator is positive off the origin, so $-1 \leq f \leq 1$.

Step 2: Endpoints and intermediate values On the x -axis, $f = 1$; on the y -axis, $f = -1$. For $-1 < c < 1$,

$$(1 - c)x^2 = (1 + c)y^2.$$

Choosing $y = 1$ and $x = \sqrt{(1 + c)/(1 - c)}$ realizes c . Hence

$$\boxed{\text{range}(f) = [-1, 1]}.$$

Step 3: Levels For $-1 < c < 1$, the level set is the pair of lines

$$\boxed{y = \pm \sqrt{\frac{1 - c}{1 + c}} x},$$

with the origin removed. At $c = 1$ and $c = -1$, the levels are the punctured x - and y -axes.

Question 7

A quadratic response has the form

$$q(x, y) = ax^2 + bxy + cy^2 + d.$$

It satisfies $q(1, 0) = 6$, $q(0, 1) = 4$, $q(1, 1) = 15$, and $q(0, 0) = 1$.

Tasks

1. Determine q .
2. Compute $q(2, -1)$.
3. Classify its level curves after removing the cross term by factoring.

Question 7 – Solution

Strategy Substitute the four samples one at a time.

Step 1: Coefficients From $q(0, 0) = 1$, $d = 1$. Then

$$a + d = 6 \implies a = 5, \quad c + d = 4 \implies c = 3.$$

Finally,

$$a + b + c + d = 15 \implies 5 + b + 3 + 1 = 15 \implies b = 6.$$

Thus $\boxed{q(x, y) = 5x^2 + 6xy + 3y^2 + 1}$.

Step 2: Evaluate $q(2, -1) = 20 - 12 + 3 + 1 = \boxed{12}$.

Step 3: Geometry Complete a square:

$$5x^2 + 6xy + 3y^2 = 5 \left(x + \frac{3}{5}y \right)^2 + \frac{6}{5}y^2.$$

Both coefficients are positive, so $q = k > 1$ gives rotated ellipses, $q = 1$ gives only the origin, and $q < 1$ gives no real points.

Question 8

Analyze

$$f(x, y) = \frac{xy}{1 + x^2 + y^2}.$$

Tasks

1. Prove strict bounds for its range.
2. Show every value between the bounds occurs.
3. Explain why the endpoints are approached but never attained.

Question 8 – Solution

Strategy Use $2|xy| \leq x^2 + y^2$ and then restrict to the lines $y = \pm x$.

Step 1: Bound Let $r^2 = x^2 + y^2$. Then

$$|f| \leq \frac{r^2/2}{1 + r^2} < \frac{1}{2}.$$

Thus the range is contained in $(-1/2, 1/2)$.

Step 2: Realize values On $y = x$,

$$f(x, x) = \frac{x^2}{1 + 2x^2},$$

which increases from 0 toward $1/2$. Solving $c = x^2/(1 + 2x^2)$ gives $x^2 = c/(1 - 2c)$ for every $0 < c < 1/2$. The line $y = -x$ similarly realizes every negative value, and $(0, 0)$ realizes 0.

Step 3: Conclusion Therefore

$$\boxed{\text{range}(f) = (-1/2, 1/2)}.$$

The denominator contains the extra positive 1, making equality in the bound impossible for finite inputs; the endpoints occur only as limits.

Question 9

Let

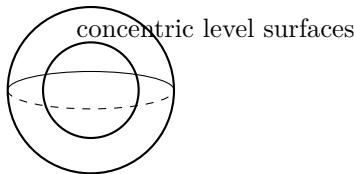
$$F(x, y, z) = \ln(4 - x^2 - y^2 - z^2).$$

Tasks

1. Find its domain and range.
2. Describe every level surface $F = c$.
3. Determine how the level radius changes as c increases.

Question 9 – Solution

Strategy Enforce positivity of the logarithm's argument and solve $F = c$ by exponentiating.



Step 1: Domain We need

$$4 - x^2 - y^2 - z^2 > 0,$$

so the domain is the open ball $x^2 + y^2 + z^2 < 4$.

Step 2: Range The logarithm's argument takes every value in $(0, 4]$. Therefore F takes every value in

$$(-\infty, \ln 4].$$

The maximum $\ln 4$ occurs at the origin; the function tends to $-\infty$ near the boundary.

Step 3: Levels From $F = c$,

$$4 - r^2 = e^c \implies r = \sqrt{4 - e^c}, \quad c \leq \ln 4.$$

These are concentric spheres, degenerating to the origin at $c = \ln 4$. As c increases, e^c increases and the radius decreases.

Question 10

An open-top box has length x , width y , and height h . A design constraint fixes the sum of these dimensions:

$$x + y + h = 12, \quad x, y, h > 0.$$

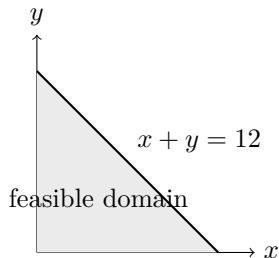
Its volume is $V(x, y) = xy(12 - x - y)$.

Tasks

1. State the natural domain.
2. Find the greatest possible volume without calculus.
3. Prove the maximizing dimensions are unique.

Question 10 – Solution

Strategy Interpret the three positive factors x , y , and $h = 12 - x - y$, then apply AM–GM.



Step 1: Domain Positivity requires

$$\boxed{x > 0, \quad y > 0, \quad x + y < 12}.$$

This is the interior of a triangular region.

Step 2: Bound The three positive numbers x, y, h have fixed sum 12. AM–GM gives

$$\frac{x + y + h}{3} \geq \sqrt[3]{xyh} \implies 4 \geq \sqrt[3]{V}.$$

Cubing yields $V \leq 64$.

Step 3: Equality Equality in AM–GM occurs exactly when

$$x = y = h = 4.$$

Thus the unique maximizing dimensions are $\boxed{x = 4, y = 4, h = 4}$ and

$$\boxed{V_{\max} = 64}.$$

Direct substitution satisfies the material constraint and confirms the volume.