

Question 1

Classify and sketch the surface

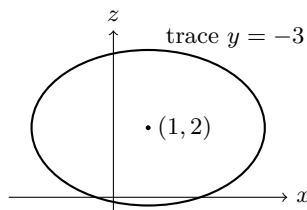
$$4x^2 + y^2 + 9z^2 - 8x + 6y - 36z + 4 = 0.$$

Tasks

1. Convert to standard form and classify it.
2. Find its center and semiaxes.
3. Derive its principal traces.
4. Find all coordinate-axis intercepts.

Question 1 – Solution

Strategy Group like variables and complete each square.



Step 1: Complete squares Write

$$4(x^2 - 2x) + (y^2 + 6y) + 9(z^2 - 4z) + 4 = 0.$$

Using $x^2 - 2x = (x - 1)^2 - 1$, $y^2 + 6y = (y + 3)^2 - 9$, and $z^2 - 4z = (z - 2)^2 - 4$ gives

$$4(x - 1)^2 + (y + 3)^2 + 9(z - 2)^2 = 45.$$

Divide by 45:

$$\frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} = 1.$$

Step 2: Geometry This is an ellipsoid centered at $(1, -3, 2)$ with semiaxes $3\sqrt{5}/2$, $3\sqrt{5}$, and $\sqrt{5}$. Its six axis endpoints relative to the center are obtained by adding and subtracting those lengths in the corresponding coordinate.

Step 3: Principal traces Through the center, the traces are

$$\begin{aligned} x = 1 : \frac{(y + 3)^2}{45} + \frac{(z - 2)^2}{5} &= 1, & y = -3 : \frac{(x - 1)^2}{45/4} + \frac{(z - 2)^2}{5} &= 1, \\ z = 2 : \frac{(x - 1)^2}{45/4} + \frac{(y + 3)^2}{45} &= 1. \end{aligned}$$

Step 4: Axis intercepts Set the other two coordinates to zero in the original equation. This gives

$$(1, 0, 0), \quad (0, -3 \pm \sqrt{5}, 0), \quad (0, 0, 2 \pm 4\sqrt{2}/3).$$

Substitution verifies each intercept and trace.

Question 2

Analyze

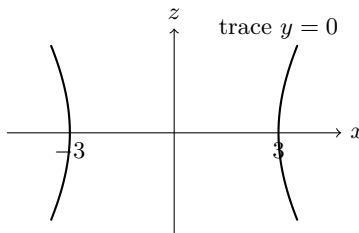
$$\frac{x^2}{9} + \frac{y^2}{4} - \frac{z^2}{16} = 1.$$

Tasks

1. Identify the surface and its axis.
2. Derive all three coordinate-plane traces.
3. Find the waist dimensions.
4. Explain why the surface is connected.

Question 2 – Solution

Strategy Read the signs and inspect traces one coordinate at a time.



Step 1: Classification Two positive squared terms and one negative term equal 1, so this is a hyperboloid of one sheet. The negative term is the z -term, hence the axis is the z -axis.

Step 2: Horizontal traces Setting $z = c$ gives

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 + \frac{c^2}{16}.$$

Every right side is positive, so every horizontal plane cuts an ellipse. At $c = 0$ the waist ellipse has semiaxes 3 and 2.

Step 3: Vertical traces At $y = 0$,

$$\frac{x^2}{9} - \frac{z^2}{16} = 1,$$

and at $x = 0$,

$$\frac{y^2}{4} - \frac{z^2}{16} = 1.$$

Both are hyperbolas opening perpendicular to the z -axis.

Conclusion Because a nonempty ellipse exists for every real $z = c$ and varies continuously with c , the surface is connected.

Question 3

For real k , classify

$$x^2 + y^2 - kz^2 = 1.$$

Tasks

1. Classify the surface for $k < 0$, $k = 0$, and $k > 0$.
2. Describe boundedness in each case.
3. Compare horizontal and vertical traces across the transition.

Question 3 – Solution

Strategy The sign of the z^2 coefficient controls the canonical type.

Case 1: $k < 0$ Write $k = -a$ with $a > 0$. Then $x^2 + y^2 + az^2 = 1$, an ellipsoid of revolution, bounded in every direction.

Case 2: $k = 0$ The equation becomes $x^2 + y^2 = 1$ with unrestricted z , so it is a circular cylinder of radius 1 parallel to the z -axis, unbounded in z .

Case 3: $k > 0$ The equation has two positive terms and one negative term, so it is a one-sheet hyperboloid about the z -axis. At $z = c$, the circle has radius $\sqrt{1 + kc^2}$; the surface is unbounded in all three coordinates.

Verification In every case the xy -trace is the unit circle. Vertical traces change from ellipses, to parallel lines, to hyperbolas, confirming the transition.

Question 4

Analyze the hyperbolic paraboloid

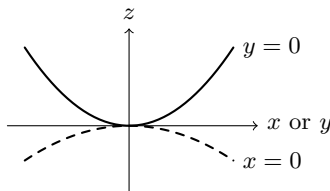
$$z = \frac{x^2}{4} - \frac{y^2}{9}.$$

Tasks

1. Derive the principal vertical traces.
2. Classify horizontal traces for $z = c$.
3. Find every line through the origin contained in the surface.
4. Justify the term “saddle.”

Question 4 – Solution

Strategy Fix one coordinate at a time, then factor the level-zero equation.



Step 1: Vertical traces If $y = 0$, then $z = x^2/4$, an upward parabola. If $x = 0$, then $z = -y^2/9$, a downward parabola. Their opposite concavities create the saddle.

Step 2: Horizontal traces At $z = c$,

$$\frac{x^2}{4} - \frac{y^2}{9} = c.$$

For $c > 0$ the hyperbola opens in the x -direction; for $c < 0$ it opens in the y -direction.

Step 3: Lines through the origin A contained line $(x, y, z) = t(a, b, c)$ must satisfy $tc = t^2(a^2/4 - b^2/9)$ for all t , so $c = 0$ and $a^2/4 = b^2/9$. Factoring gives

$$\left(\frac{x}{2} - \frac{y}{3}\right)\left(\frac{x}{2} + \frac{y}{3}\right) = 0.$$

Thus the two lines are $\boxed{z = 0, y = \pm 3x/2}$. Direct substitution confirms every point on each line lies on the surface.

Question 5

A centered quadric has horizontal traces only for $|z| \geq 3$, where

$$x^2 + y^2 = 4 \left(\frac{z^2}{9} - 1 \right).$$

Tasks

1. Reconstruct and classify the quadric.
2. Find its vertices and axis.
3. Derive both principal vertical traces.
4. Explain the empty central region.

Question 5 – Solution

Strategy Rearrange the trace family and interpret both its signs and its existence condition.

Step 1: Reconstruct Move terms and divide by 4:

$$x^2 + y^2 = \frac{4}{9}z^2 - 4 \implies \boxed{\frac{z^2}{9} - \frac{x^2}{4} - \frac{y^2}{4} = 1}.$$

Step 2: Geometry One positive square and two negative squares equal 1, so this is a two-sheet hyperboloid with z -axis. Setting $x = y = 0$ gives vertices $\boxed{(0, 0, \pm 3)}$.

Step 3: Vertical traces Setting $y = 0$ and $x = 0$ gives

$$\boxed{\frac{z^2}{9} - \frac{x^2}{4} = 1}, \quad \boxed{\frac{z^2}{9} - \frac{y^2}{4} = 1}.$$

Step 4: Gap Since $x^2 + y^2 \geq 0$, the given trace requires $z^2/9 - 1 \geq 0$, hence $|z| \geq 3$. No surface points lie in the central slab $-3 < z < 3$.

Question 6

Find and describe the intersection of

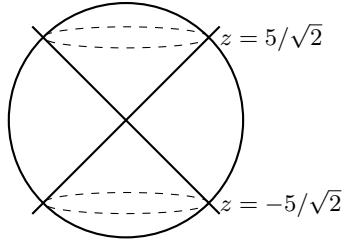
$$x^2 + y^2 + z^2 = 25 \quad \text{and} \quad z^2 = x^2 + y^2.$$

Tasks

1. Determine every component of the intersection.
2. Find each component's exact height and radius.
3. Verify the result in both original equations.

Question 6 – Solution

Strategy Substitute the cone relation into the sphere, then retain both signs of z .



Step 1: Solve for height Since $x^2 + y^2 = z^2$, the sphere gives

$$z^2 + z^2 = 25 \implies 2z^2 = 25 \implies z = \pm \frac{5}{\sqrt{2}}.$$

Step 2: Solve for radius On either level,

$$x^2 + y^2 = z^2 = \frac{25}{2}.$$

Thus the intersection consists of two circles:

$$\boxed{z = \pm \frac{5}{\sqrt{2}}, \quad x^2 + y^2 = \frac{25}{2}}.$$

Each has radius $\boxed{5/\sqrt{2}}$. Both signs are essential because the cone is double. Substitution verifies both circles in both original equations.

Question 7

The ellipsoid

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

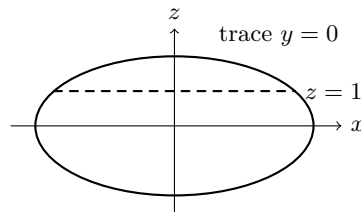
is cut by the plane $z = 1$.

Tasks

1. Find the section in standard form.
2. Determine its semiaxes and area.
3. Compare it quantitatively with the central section $z = 0$.

Question 7 – Solution

Strategy Substitute the fixed height and rewrite the remaining equation in standard ellipse form.



Step 1: Substitute With $z = 1$,

$$\frac{x^2}{16} + \frac{y^2}{9} + \frac{1}{4} = 1 \implies \frac{x^2}{16} + \frac{y^2}{9} = \frac{3}{4}.$$

Divide by $3/4$:

$$\boxed{\frac{x^2}{12} + \frac{y^2}{27/4} = 1}.$$

The semiaxes are $\boxed{2\sqrt{3}}$ and $\boxed{3\sqrt{3}/2}$.

Step 2: Area Therefore

$$A = \pi ab = \pi(2\sqrt{3})(3\sqrt{3}/2) = \boxed{9\pi}.$$

Comparison At $z = 0$, the semiaxes are 4 and 3, so the central area is 12π . The section at $z = 1$ has area $3/4$ as large, matching the right side $3/4$ before normalization.

Question 8

An elliptic paraboloid has vertex $(1, -2, 3)$, axis parallel to the z -axis, opens downward, and its trace in $z = -1$ is

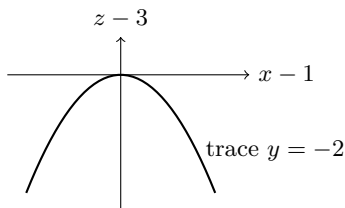
$$\frac{(x-1)^2}{8} + \frac{(y+2)^2}{4} = 1.$$

Tasks

1. Reconstruct the surface equation.
2. Find the trace at $z = 2$ in standard form.
3. State that trace's semiaxes and verify the supplied trace.

Question 8 – Solution

Strategy Use translated form $z - 3 = -A(x - 1)^2 - B(y + 2)^2$ and fit the known level.



Step 1: Use the known trace At $z = -1$, the vertical drop is 4. The given ellipse is equivalent to

$$\frac{1}{2}(x - 1)^2 + (y + 2)^2 = 4.$$

Hence the surface equation is

$$z - 3 = -\frac{1}{2}(x - 1)^2 - (y + 2)^2.$$

Step 2: New trace Set $z = 2$:

$$1 = \frac{1}{2}(x - 1)^2 + (y + 2)^2,$$

or

$$\frac{(x - 1)^2}{2} + (y + 2)^2 = 1.$$

Its semiaxes are $\sqrt{2}$ and 1. Substituting $z = -1$ back into the derived surface recovers the supplied ellipse.

Question 9

Classify the real point sets

$$x^2 + y^2 + z^2 - 2xy - 2xz + 2yz = c$$

Tasks

1. Factor the quadratic expression.
2. Classify the real point set for $c < 0$, $c = 0$, and $c > 0$.
3. Explain the degeneration geometrically.

Question 9 – Solution

Strategy Test whether the quadratic expression is a perfect square.

Step 1: Factor Expanding

$$(x - y - z)^2 = x^2 + y^2 + z^2 - 2xy - 2xz + 2yz.$$

Thus the equation is simply

$$(x - y - z)^2 = c.$$

Case analysis If $c < 0$, no real square can equal c , so the set is empty. If $c = 0$, then $x - y - z = 0$, a single plane. If $c > 0$, taking square roots gives

$$x - y - z = \pm\sqrt{c},$$

so the set is two distinct parallel planes.

Conclusion The classification is $\boxed{\emptyset}$, $\boxed{\text{one plane}}$, and $\boxed{\text{two parallel planes}}$ for the three cases. Substitution into the factored form verifies each result.

Question 10

Consider

$$5x^2 - 6xy + 5y^2 - z^2 = 0, \quad u = \frac{x+y}{\sqrt{2}}, \quad v = \frac{x-y}{\sqrt{2}}.$$

Tasks

1. Rewrite the equation in u, v, z .
2. Classify the surface and identify its axis.
3. Derive the traces in $z = c$, $u = 0$, and $v = 0$.
4. Relate the principal directions to the original xy -axes.

Question 10 – Solution

Strategy Express $x^2 + y^2$ and xy in rotated coordinates so the cross term disappears.

Step 1: Invert and substitute We have

$$x = \frac{u+v}{\sqrt{2}}, \quad y = \frac{u-v}{\sqrt{2}}, \quad x^2 + y^2 = u^2 + v^2, \quad xy = \frac{u^2 - v^2}{2}.$$

Therefore

$$5(u^2 + v^2) - 3(u^2 - v^2) - z^2 = 0, \quad \boxed{2u^2 + 8v^2 - z^2 = 0}.$$

Step 2: Classify This is an elliptic cone with vertex at the origin and axis the z -axis.

Step 3: Traces For $z = c \neq 0$,

$$\boxed{\frac{u^2}{c^2/2} + \frac{v^2}{c^2/8} = 1}.$$

For $v = 0$, $z = \pm\sqrt{2}u$; for $u = 0$, $z = \pm 2\sqrt{2}v$. At $z = 0$, only the vertex remains.

Step 4: Interpret rotation The u -axis is $x = y$ and the v -axis is $x = -y$, so the principal horizontal directions are rotated 45° from the original axes.