

Question 1

A plane passes through

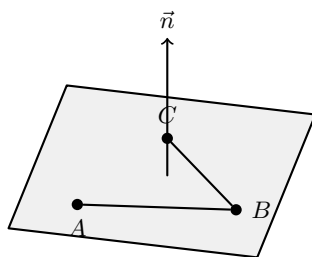
$$A = (1, 0, 2), \quad B = (3, 1, -1), \quad C = (-1, 4, 0).$$

Tasks

1. Derive a Cartesian equation of the plane from two in-plane vectors.
2. Write a unit normal vector.
3. Find all three coordinate-axis intercepts and verify the three given points.

Question 1 – Solution

Strategy Form $\overrightarrow{AB}$ and $\overrightarrow{AC}$; their cross product is normal to the plane.



Step 1: In-plane vectors Subtract coordinates component by component:

$$\begin{aligned}\overrightarrow{AB} &= B - A = \langle 3 - 1, 1 - 0, -1 - 2 \rangle = \langle 2, 1, -3 \rangle, \\ \overrightarrow{AC} &= C - A = \langle -1 - 1, 4 - 0, 0 - 2 \rangle = \langle -2, 4, -2 \rangle.\end{aligned}$$

Step 2: Normal vector Their cross product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1(-2) - (-3)(4), (-3)(-2) - 2(-2), 2(4) - 1(-2) \rangle = \langle 10, 10, 10 \rangle.$$

Divide by 10 to use the primitive normal $\vec{n} = \langle 1, 1, 1 \rangle$. The point-normal equation through A is

$$(x - 1) + (y - 0) + (z - 2) = 0.$$

Therefore

$$\boxed{x + y + z = 3}, \quad \boxed{\hat{n} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle}.$$

Step 3: Intercepts On the x -axis, $y = z = 0$, so $x = 3$. The other axes work similarly:

$$\boxed{(3, 0, 0), (0, 3, 0), (0, 0, 3)}.$$

Verification Substitution of A , B , and C gives 3 each time. The two in-plane vectors dot to zero with $\langle 1, 1, 1 \rangle$, independently confirming the orientation.

Question 2

A plane contains the line

$$L : \vec{r} = \langle 1, -2, 0 \rangle + t \langle 2, 1, -1 \rangle$$

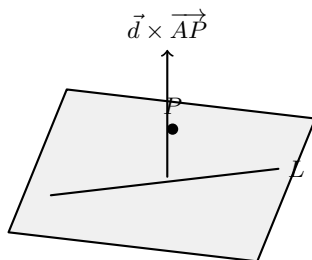
and the point $P = (0, 1, 4)$.

Tasks

1. Find a Cartesian equation of the plane.
2. Explain why the data determine a unique plane.
3. Verify that the entire line and the point lie in the plane.

Question 2 – Solution

Strategy Combine the line direction with a vector from a point on the line to P . Their cross product is a plane normal.



Step 1: Two directions in the plane Take $A = (1, -2, 0)$ from the line. Its direction is $\vec{d} = \langle 2, 1, -1 \rangle$, while

$$\overrightarrow{AP} = P - A = \langle 0 - 1, 1 - (-2), 4 - 0 \rangle = \langle -1, 3, 4 \rangle.$$

Step 2: Find a normal Compute

$$\vec{d} \times \overrightarrow{AP} = \langle 1(4) - (-1)(3), (-1)(-1) - 2(4), 2(3) - 1(-1) \rangle = \langle 7, -7, 7 \rangle.$$

Thus $\vec{n} = \langle 1, -1, 1 \rangle$. Using point A ,

$$(x - 1) - (y + 2) + (z - 0) = 0,$$

so the plane is

$$\boxed{x - y + z = 3}.$$

Uniqueness The vectors $\vec{d}$ and $\overrightarrow{AP}$ are not parallel, so they span exactly one plane.

Verification Point P gives $0 - 1 + 4 = 3$. A general point on L is $(1 + 2t, -2 + t, -t)$, and

$$(1 + 2t) - (-2 + t) - t = 3$$

for every t , proving that the whole line is contained in the plane.

Question 3

For a real parameter k , let

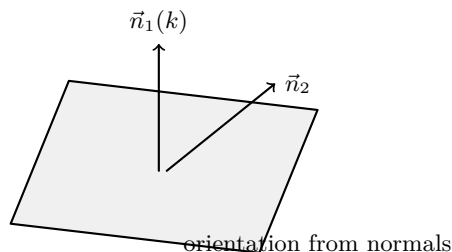
$$\Pi_k : kx + 2y - z = 5, \quad \Sigma : 2x - y + 2z = 7.$$

Tasks

1. Find k if the planes are perpendicular.
2. Determine whether any k makes the planes parallel.
3. Find every k for which the acute angle between the planes is 60° .

Question 3 – Solution

Strategy Plane orientation is controlled by the normals $\vec{n}_k = \langle k, 2, -1 \rangle$ and $\vec{m} = \langle 2, -1, 2 \rangle$.



Step 1: Perpendicularity Calculate

$$\vec{n}_k \cdot \vec{m} = k(2) + 2(-1) + (-1)(2) = 2k - 4.$$

Perpendicular normals require $2k - 4 = 0$, hence $\boxed{k = 2}$.

Step 2: Parallelism Parallelism would require $\langle k, 2, -1 \rangle = \lambda \langle 2, -1, 2 \rangle$. From $2 = -\lambda$, $\lambda = -2$; but the third coordinate would require $-1 = 2(-2) = -4$, a contradiction. Hence $\boxed{\text{no real } k \text{ makes them parallel}}$.

Step 3: The 60° condition The angle formula gives

$$\frac{|2k - 4|}{3\sqrt{k^2 + 5}} = \frac{1}{2}.$$

Cross-multiply and square:

$$2|2k - 4| = 3\sqrt{k^2 + 5} \implies 16(k - 2)^2 = 9(k^2 + 5).$$

Expanding gives $16k^2 - 64k + 64 = 9k^2 + 45$, so

$$7k^2 - 64k + 19 = 0.$$

The quadratic formula yields

$$k = \frac{64 \pm \sqrt{64^2 - 4(7)(19)}}{14} = \frac{64 \pm 18\sqrt{11}}{14},$$

and therefore

$$\boxed{k = \frac{32 \pm 9\sqrt{11}}{7}}.$$

Verification Neither root makes the original denominator zero, and substitution into the unsquared equation gives the positive value $1/2$, so no extraneous root remains.

Question 4

Consider the planes

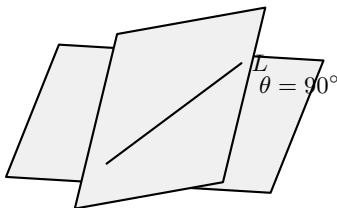
$$\Pi_1 : x + y - z = 2, \quad \Pi_2 : 2x - y + z = 1.$$

Tasks

1. Find a vector and parametric equation for their line of intersection.
2. Find the dihedral angle between the planes.
3. Verify the line equation in both plane equations.

Question 4 – Solution

Strategy The intersection direction is perpendicular to both normals. Solve the two plane equations for one point and one free parameter.



Step 1: Solve simultaneously Add the equations:

$$(x + y - z) + (2x - y + z) = 2 + 1 \implies 3x = 3 \implies x = 1.$$

Substitute $x = 1$ into $x + y - z = 2$:

$$1 + y - z = 2 \implies y = 1 + z.$$

Let the free coordinate be $z = t$. Then $y = 1 + t$, so

$$\boxed{\vec{r} = \langle 1, 1, 0 \rangle + t \langle 0, 1, 1 \rangle}.$$

Step 2: Confirm the direction The normals are $\vec{n}_1 = \langle 1, 1, -1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 1 \rangle$. Their cross product is $\langle 0, -3, -3 \rangle$, parallel to $\langle 0, 1, 1 \rangle$.

Step 3: Angle Compute $\vec{n}_1 \cdot \vec{n}_2 = 2 - 1 - 1 = 0$. Therefore

$$\cos \theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\| \|\vec{n}_2\|} = 0, \quad \boxed{\theta = 90^\circ}.$$

Verification Substitution of $(1, 1 + t, t)$ gives 2 in Π_1 and 1 in Π_2 for every t .

Question 5

Let $P = (4, -1, 5)$ and

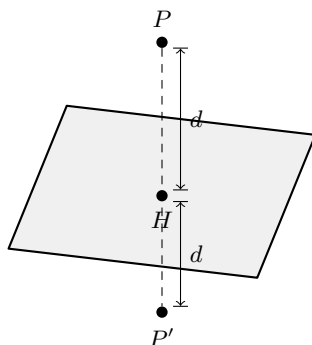
$$\Pi : 2x - y + 2z = 3.$$

Tasks

1. Find the perpendicular projection H of P onto Π .
2. Find the distance from P to Π .
3. Reflect P across Π and find the image point P' .
4. Verify the midpoint and perpendicularity conditions.

Question 5 – Solution

Strategy Move from P along the normal $\vec{n} = \langle 2, -1, 2 \rangle$. The signed equation value determines the required multiple.



Step 1: Signed displacement Write the plane as $F(x, y, z) = 2x - y + 2z - 3 = 0$. At P ,

$$2(4) - (-1) + 2(5) - 3 = 16, \quad \|\vec{n}\|^2 = 9.$$

Step 2: Projection A point on the perpendicular through P is $P - s\vec{n}$. To reach the plane,

$$F(P - s\vec{n}) = F(P) - s\|\vec{n}\|^2 = 16 - 9s = 0,$$

so $s = 16/9$. Hence

$$H = P - \frac{16}{9}\vec{n} = \left(4 - \frac{32}{9}, -1 + \frac{16}{9}, 5 - \frac{32}{9}\right) = \boxed{\left(\frac{4}{9}, \frac{7}{9}, \frac{13}{9}\right)}.$$

Step 3: Distance Since $P - H = (16/9)\vec{n}$,

$$d(P, \Pi) = \frac{16}{9}\|\vec{n}\| = \frac{16}{9}(3), \quad \boxed{d(P, \Pi) = \frac{16}{3}}.$$

Step 4: Reflection Since H is the midpoint of PP' ,

$$P' = 2H - P = P - \frac{32}{9}\vec{n} = \boxed{\left(-\frac{28}{9}, \frac{23}{9}, -\frac{19}{9}\right)}.$$

Verification Substituting H into the plane gives 3. Also $P - H = (16/9)\vec{n}$, so PH is perpendicular to the plane, and direct averaging of P and P' gives H .

Question 6

Among all points on the plane

$$\Pi : x + 2y + 2z = 9,$$

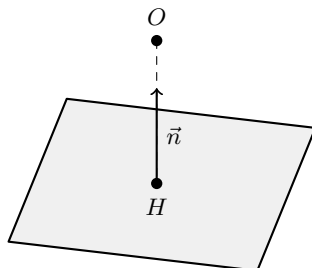
find the point closest to the origin.

Tasks

1. Determine the closest point and minimum distance.
2. Prove global minimality without using calculus.
3. Find the sphere centered at the origin tangent to Π .

Question 6 – Solution

Strategy The shortest segment from a point to a plane follows the plane's normal. Use Cauchy–Schwarz for a global proof.



Step 1: Follow the normal The normal is $\vec{n} = \langle 1, 2, 2 \rangle$ and

$$\|\vec{n}\| = \sqrt{1^2 + 2^2 + 2^2} = 3.$$

A point on the normal line through the origin has form $(t, 2t, 2t)$. Imposing the plane equation gives

$$t + 2(2t) + 2(2t) = 9 \implies 9t = 9 \implies t = 1.$$

Hence

$$\boxed{H = (1, 2, 2)}, \quad \boxed{d(O, \Pi) = \|H\| = 3}.$$

Step 2: Global proof For any $X = (x, y, z)$ on Π , $\vec{n} \cdot X = 9$. Cauchy–Schwarz gives

$$9 = |\vec{n} \cdot X| \leq \|\vec{n}\| \|X\| = 3\|X\|,$$

so $\|X\| \geq 3$. Equality occurs precisely when X is parallel to $\vec{n}$, as H is.

Step 3: Tangent sphere A sphere centered at the origin and reaching H has radius 3, so its equation is

$$\boxed{x^2 + y^2 + z^2 = 9}.$$

Its radius to H is normal to Π , so the plane is tangent there.

Question 7

Let

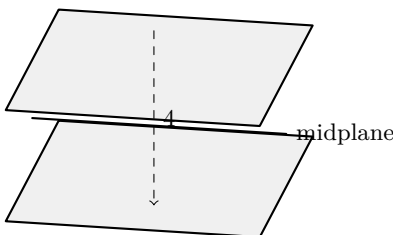
$$\Pi_1 : 2x - 2y + z = 7, \quad \Pi_2 : 2x - 2y + z = -5.$$

Tasks

1. Find the distance between the parallel planes.
2. Derive the plane exactly halfway between them.
3. Find endpoints of one shortest segment joining the planes and verify its midpoint lies on the halfway plane.

Question 7 – Solution

Strategy Put the equations over the same normal. Their constant difference divided by the normal length is the separation.



Step 1: Distance Both planes have normal $\vec{n} = \langle 2, -2, 1 \rangle$, whose length is

$$\|\vec{n}\| = \sqrt{2^2 + (-2)^2 + 1^2} = 3.$$

Because their left sides are identical, the distance is the absolute difference of constants divided by $\|\vec{n}\|$:

$$d(\Pi_1, \Pi_2) = \frac{|7 - (-5)|}{3} = 4.$$

Step 2: Halfway plane A parallel plane has form $2x - 2y + z = c$. Equal distance from c to 7 and -5 requires

$$7 - c = c - (-5) \implies 2c = 2 \implies c = 1.$$

Thus

$$2x - 2y + z = 1.$$

Step 3: Shortest connector Choose $A = (3, 0, 1)$; indeed $2(3) - 2(0) + 1 = 7$. The unit normal is $\vec{n}/3$, so moving four units toward Π_2 gives

$$B = A - \frac{4}{3} \langle 2, -2, 1 \rangle = \left(\frac{1}{3}, \frac{8}{3}, -\frac{1}{3} \right).$$

Substitution gives $2(1/3) - 2(8/3) - 1/3 = -5$, so $B \in \Pi_2$. Its midpoint with A is

$$M = \left(\frac{5}{3}, \frac{4}{3}, \frac{1}{3} \right).$$

Verification $2(5/3) - 2(4/3) + 1/3 = 1$. Also $B - A$ is parallel to the common normal, so AB is perpendicular to both planes and is therefore a shortest connector.

Question 8

All planes containing the intersection of

$$\Pi_1 : x + y + z = 1 \quad \text{and} \quad \Pi_2 : x - y + 2z = 3$$

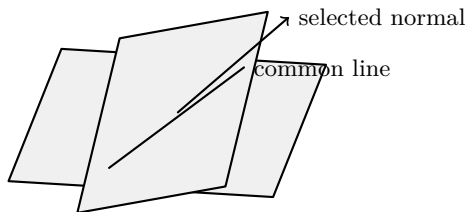
form a pencil of planes.

Tasks

1. Write a one-parameter equation for this pencil.
2. Find the member perpendicular to $\Sigma : x + 2y - z = 4$.
3. Verify that the selected plane contains the entire common line of Π_1 and Π_2 .

Question 8 – Solution

Strategy Every combination $F_1 + \lambda F_2 = 0$ vanishes wherever both original plane expressions vanish. Choose λ using normal-vector perpendicularity.



Step 1: Build the pencil Move all terms to the left:

$$F_1 = x + y + z - 1, \quad F_2 = x - y + 2z - 3.$$

At every point on the common line, $F_1 = F_2 = 0$. Therefore every equation

$$F_1 + \lambda F_2 = 0 \ (\lambda \in \mathbb{R}), \quad \text{together with } F_2 = 0$$

contains that line. The member $F_2 = 0$ is also needed for the full pencil; its normal has dot product -3 with $\langle 1, 2, -1 \rangle$, so it is not the requested perpendicular member. Collecting coefficients shows its normal is

$$\vec{n}_\lambda = \langle 1 + \lambda, 1 - \lambda, 1 + 2\lambda \rangle.$$

Step 2: Impose perpendicularity The normal of Σ is $\vec{m} = \langle 1, 2, -1 \rangle$. Perpendicular planes have perpendicular normals, so

$$\vec{n}_\lambda \cdot \langle 1, 2, -1 \rangle = 2 - 3\lambda = 0,$$

because $(1 + \lambda) + 2(1 - \lambda) - (1 + 2\lambda) = 2 - 3\lambda$. Thus $\lambda = 2/3$. Multiplying by 3 gives

$$3F_1 + 2F_2 = 0,$$

and expanding yields

$$5x + y + 7z = 9.$$

Verification Any point common to Π_1 and Π_2 has $F_1 = F_2 = 0$, so it satisfies $3F_1 + 2F_2 = 0$. Its normal $\langle 5, 1, 7 \rangle$ dots with $\langle 1, 2, -1 \rangle$ to 0.

Question 9

Find the locus of points equidistant from the intersecting planes

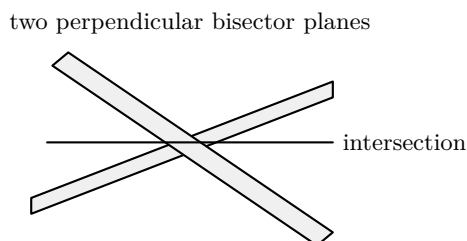
$$\Pi_1 : x - 2y + 2z - 3 = 0, \quad \Pi_2 : 2x + 2y - z + 6 = 0.$$

Tasks

1. Derive equations for both components of the locus.
2. Explain why they are angle-bisector planes.
3. Prove that the two resulting planes are perpendicular.

Question 9 – Solution

Strategy Use the point-to-plane distance formula with absolute values. Equal normal lengths simplify the two sign cases.



Step 1: Distance equation The normal lengths are

$$\sqrt{1^2 + (-2)^2 + 2^2} = 3, \quad \sqrt{2^2 + 2^2 + (-1)^2} = 3.$$

Thus equality of point-to-plane distances becomes

$$|x - 2y + 2z - 3| = |2x + 2y - z + 6|.$$

Step 2: Resolve the absolute values Let the two expressions be F_1 and F_2 . The equation $|F_1| = |F_2|$ gives two cases.

If $F_1 = F_2$, then

$$x - 2y + 2z - 3 = 2x + 2y - z + 6 \implies x + 4y - 3z + 9 = 0.$$

If $F_1 = -F_2$, then

$$x - 2y + 2z - 3 = -2x - 2y + z - 6 \implies 3x + z + 3 = 0.$$

Therefore the locus consists of

$$\boxed{x + 4y - 3z + 9 = 0}, \quad \boxed{3x + z + 3 = 0}.$$

Interpretation On either plane, the signed perpendicular displacements from Π_1 and Π_2 have equal magnitude. These are precisely the internal and external angle-bisector planes, and both contain the original planes' intersection line.

Perpendicularity Normals may be chosen as $\langle 1, 4, -3 \rangle$ and $\langle 3, 0, 1 \rangle$. Their dot product is $3 - 3 = 0$, so the bisector planes are perpendicular.

Verification Substituting any point from either boxed plane into the absolute-value equation recovers equal distances.

Question 10

A plane cuts the positive coordinate axes at A , B , and C . The centroid of the intercept triangle ABC is

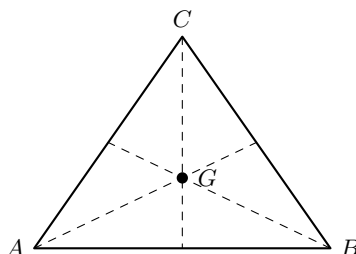
$$G = (2, 3, 4).$$

Tasks

1. Determine A , B , and C .
2. Find the plane in intercept and Cartesian forms.
3. Find the area of triangle ABC .
4. Find the volume of the tetrahedron bounded by the plane and the coordinate planes.

Question 10 – Solution

Strategy Write $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Their centroid directly determines the three intercepts.



Step 1: Recover the intercepts Write $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Then

$$G = \frac{A + B + C}{3} = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right),$$

so $a/3 = 2$, $b/3 = 3$, and $c/3 = 4$. Hence

$$\boxed{A = (6, 0, 0), \quad B = (0, 9, 0), \quad C = (0, 0, 12)}.$$

Step 2: Plane equation A plane with intercepts a, b, c has equation $x/a + y/b + z/c = 1$. Therefore

$$\boxed{\frac{x}{6} + \frac{y}{9} + \frac{z}{12} = 1}, \quad \boxed{6x + 4y + 3z = 36}.$$

Step 3: Triangle area Compute

$$\begin{aligned} \overrightarrow{AB} &= \langle -6, 9, 0 \rangle, & \overrightarrow{AC} &= \langle -6, 0, 12 \rangle, \\ \overrightarrow{AB} \times \overrightarrow{AC} &= \langle 9(12), 0(-6) - (-6)(12), (-6)(0) - 9(-6) \rangle = \langle 108, 72, 54 \rangle. \end{aligned}$$

Its length is

$$\sqrt{108^2 + 72^2 + 54^2} = \sqrt{19764} = 18\sqrt{61}.$$

Hence $\boxed{\text{Area}(ABC) = 9\sqrt{61}}.$

Step 4: Tetrahedron volume The coordinate-axis edges from the origin are mutually perpendicular with lengths 6, 9, 12. A tri-rectangular tetrahedron has one-sixth the corresponding box volume, so

$$\boxed{V = \frac{1}{6}(6)(9)(12) = 108}.$$

Verification Substitution of each intercept into $6x + 4y + 3z = 36$ succeeds, and their coordinate average is $(2, 3, 4)$.