

Question 1

A line passes through the points

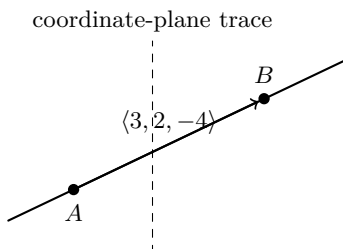
$$A = (-2, 1, 3), \quad B = (4, 5, -5).$$

Tasks

1. Write the line in vector, parametric, and symmetric forms using a primitive integer direction vector.
2. Find the line's intersection with each coordinate plane.
3. Verify that the three plane-intersection points lie on your line.

Question 1 – Solution

Strategy Subtract the given points to obtain a direction, reduce it, and set one coordinate at a time equal to zero to find the coordinate-plane traces.



Line equations Since $B - A = \langle 6, 4, -8 \rangle = 2 \langle 3, 2, -4 \rangle$,

$$\vec{r} = \langle -2, 1, 3 \rangle + t \langle 3, 2, -4 \rangle,$$

$$x = -2 + 3t, \quad y = 1 + 2t, \quad z = 3 - 4t,$$

$$\frac{x + 2}{3} = \frac{y - 1}{2} = \frac{z - 3}{-4}.$$

Coordinate-plane traces Setting $z = 0$, $x = 0$, and $y = 0$ gives, respectively,

$$L \cap xy = \left(\frac{1}{4}, \frac{5}{2}, 0 \right), \quad L \cap yz = \left(0, \frac{7}{3}, \frac{1}{3} \right), \quad L \cap xz = \left(-\frac{7}{2}, 0, 5 \right).$$

Verification These correspond to $t = 3/4$, $2/3$, and $-1/2$ in the same parametric equations, so all three points lie on the line.

Question 2

Consider the line

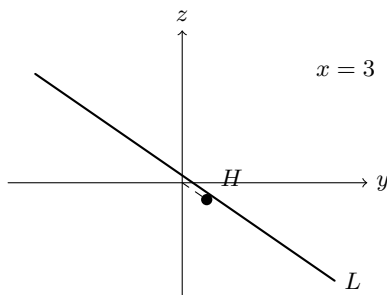
$$L : \quad x = 3, \quad \frac{y+2}{4} = \frac{z-1}{-3}.$$

Tasks

1. Write vector and parametric equations for L .
2. Explain geometrically what the equation $x = 3$ says about the line.
3. Find the point on L closest to the origin and determine that minimum distance.

Question 2 – Solution

Strategy Use the common symmetric ratio as a parameter. For the closest point, minimize the squared distance to avoid unnecessary square roots.



Representations Let the common ratio equal t . Then

$$\vec{r} = \langle 3, -2, 1 \rangle + t \langle 0, 4, -3 \rangle, \quad x = 3, \quad y = -2 + 4t, \quad z = 1 - 3t.$$

The zero x -component of the direction means the entire line lies in the plane $x = 3$ and is parallel to the yz -plane.

Closest point The squared distance from the origin is

$$D^2(t) = 9 + (-2 + 4t)^2 + (1 - 3t)^2 = 25t^2 - 22t + 14.$$

Its minimum occurs at $t = 22/50 = 11/25$. Hence

$$H = \left(3, -\frac{6}{25}, -\frac{8}{25} \right), \quad d(O, L) = \frac{\sqrt{229}}{5}.$$

Verification The vector H from the origin has dot product 0 with $\langle 0, 4, -3 \rangle$, confirming that OH is perpendicular to L .

Question 3

Points

$$A = (1, -2, 3), \quad B = (5, 4, -1), \quad P = (4, u, v)$$

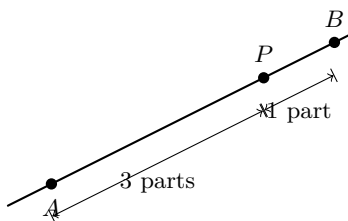
are collinear, and P lies between A and B .

Tasks

1. Determine u and v .
2. Write a parametric equation of the common line.
3. Find the ratio $AP : PB$ and verify it using distances.

Question 3 – Solution

Strategy Parameterize the segment from A to B . The known x -coordinate of P determines the parameter, after which the other coordinates and division ratio follow.



Parameter and coordinates Since $B - A = \langle 4, 6, -4 \rangle$,

$$\vec{r}(t) = \langle 1, -2, 3 \rangle + t \langle 4, 6, -4 \rangle.$$

At P , $1 + 4t = 4$, so $t = 3/4$. Therefore

$$u = -2 + 6\left(\frac{3}{4}\right) = \frac{5}{2}, \quad v = 3 - 4\left(\frac{3}{4}\right) = 0,$$

and

$$\boxed{P = \left(4, \frac{5}{2}, 0\right)}.$$

Division ratio The parameter places P three-fourths of the way from A to B , so

$$\boxed{AP : PB = 3 : 1}.$$

Verification We have $P - A = \frac{3}{4}(B - A)$ and $B - P = \frac{1}{4}(B - A)$; taking lengths preserves the ratio $3 : 1$.

Question 4

For a real parameter k , consider

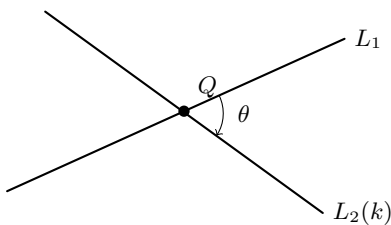
$$\begin{aligned}L_1 : \vec{r} &= \langle 1, 2, 0 \rangle + s \langle 2, -1, 3 \rangle, \\L_2(k) : \vec{r} &= \langle 4, -2, 7 \rangle + t \langle 1, k, -1 \rangle.\end{aligned}$$

Tasks

1. Find the value of k for which the lines intersect.
2. Find the intersection point.
3. For that value of k , find the acute angle between the lines.

Question 4 – Solution

Strategy The x - and z -coordinate equations determine s and t without k ; the y -coordinate then determines the required parameter.



Intersection Equating x - and z -coordinates gives

$$1 + 2s = 4 + t, \quad 3s = 7 - t.$$

The unique solution is $s = 2$, $t = 1$. The y -coordinates then require

$$2 - s = -2 + kt \implies 0 = -2 + k,$$

so $k = 2$. Both lines meet at $Q = (5, 0, 6)$.

Angle The directions are $\vec{d}_1 = \langle 2, -1, 3 \rangle$ and $\vec{d}_2 = \langle 1, 2, -1 \rangle$. Thus

$$\cos \theta = \frac{|\vec{d}_1 \cdot \vec{d}_2|}{\|\vec{d}_1\| \|\vec{d}_2\|} = \frac{3}{\sqrt{14}\sqrt{6}} = \frac{3}{2\sqrt{21}}.$$

Hence $\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right) \approx 70.9^\circ$.

Verification Substitution of $s = 2$, $t = 1$, and $k = 2$ produces $(5, 0, 6)$ in both equations.

Question 5

Let

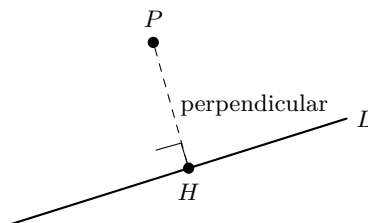
$$P = (4, 3, -1), \quad L : \vec{r} = \langle 1, -1, 2 \rangle + t \langle 2, 1, 2 \rangle.$$

Tasks

1. Find the point H on L closest to P .
2. Write an equation of the line through P that intersects L perpendicularly at H .
3. Find the distance from P to L .

Question 5 – Solution

Strategy At the closest point, the displacement from the line to P is perpendicular to the line's direction vector.



Perpendicular foot Write $A = (1, -1, 2)$ and $\vec{d} = \langle 2, 1, 2 \rangle$. For $H = A + t\vec{d}$,

$$(P - H) \cdot \vec{d} = 0 \implies (P - A - t\vec{d}) \cdot \vec{d} = 0.$$

Since $(P - A) \cdot \vec{d} = 4$ and $\vec{d} \cdot \vec{d} = 9$, $t = 4/9$. Therefore

$$H = \left(\frac{17}{9}, -\frac{5}{9}, \frac{26}{9} \right).$$

Perpendicular line and distance A direction from P to H is $\langle -19, -32, 35 \rangle$ after multiplying by 9. Thus

$$\vec{r} = \langle 4, 3, -1 \rangle + u \langle -19, -32, 35 \rangle.$$

Also,

$$d(P, L) = \|P - H\| = \frac{\sqrt{2610}}{9} = \frac{\sqrt{290}}{3}.$$

Verification $\langle -19, -32, 35 \rangle \cdot \langle 2, 1, 2 \rangle = -38 - 32 + 70 = 0$, and H satisfies the original line equation.

Question 6

Consider the two lines

$$L_1 : \vec{r} = s \langle 1, 2, 0 \rangle ,$$

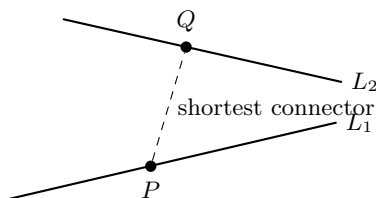
$$L_2 : \vec{r} = \langle 1, 0, 3 \rangle + t \langle 2, -1, 1 \rangle .$$

Tasks

1. Show that the lines are skew.
2. Find the point on each line at which their shortest connecting segment begins and ends.
3. Determine the distance between the lines.

Question 6 – Solution

Strategy For closest points $P \in L_1$ and $Q \in L_2$, the connector $Q - P$ must be perpendicular to both line directions.



Classification The directions $\langle 1, 2, 0 \rangle$ and $\langle 2, -1, 1 \rangle$ are not parallel. Solving the first two intersection equations gives $s = 1/5$, $t = -2/5$, but the third coordinates are then 0 and $13/5$, so the lines do not intersect. They are skew.

Closest points Let

$$P = (s, 2s, 0), \quad Q = (1 + 2t, -t, 3 + t).$$

The conditions $(Q - P) \cdot \langle 1, 2, 0 \rangle = 0$ and $(Q - P) \cdot \langle 2, -1, 1 \rangle = 0$ yield

$$s = \frac{1}{5}, \quad t = -\frac{5}{6}.$$

Thus

$$\boxed{P = \left(\frac{1}{5}, \frac{2}{5}, 0\right)}, \quad \boxed{Q = \left(-\frac{2}{3}, \frac{5}{6}, \frac{13}{6}\right)}.$$

Distance and verification Since $Q - P = \frac{13}{30} \langle -2, 1, 5 \rangle$,

$$\boxed{d(L_1, L_2) = \frac{13\sqrt{30}}{30} = \frac{13}{\sqrt{30}}}.$$

The vector $\langle -2, 1, 5 \rangle$ has zero dot product with both line directions.

Question 7

A line through

$$P = (-1, 2, 0)$$

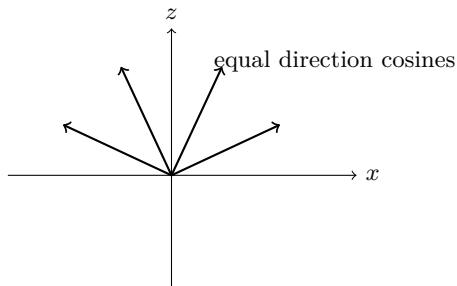
makes equal acute angles with the x -, y -, and z -axes (treated as unoriented lines). Here the first octant includes its boundary planes.

Tasks

1. Find all geometrically distinct lines through P having this property.
2. Determine which of these lines contain at least one point in the first octant.
3. State the parameter intervals corresponding to the portions lying in the first octant.

Question 7 – Solution

Strategy Equal inclination means the three direction cosines have equal absolute value. Direction vectors that differ only by an overall nonzero factor define the same line.



All lines Up to reversal, there are four sign patterns. The distinct lines are

$$\vec{r} = \langle -1, 2, 0 \rangle + t\vec{d},$$

where

$$\vec{d} \in \{ \langle 1, 1, 1 \rangle, \langle 1, 1, -1 \rangle, \langle 1, -1, 1 \rangle, \langle 1, -1, -1 \rangle \}.$$

Each direction has direction cosines with absolute value $1/\sqrt{3}$.

First-octant test For $\langle 1, 1, 1 \rangle$, the coordinates are $(-1 + t, 2 + t, t)$, all nonnegative when $t \geq 1$. For $\langle 1, -1, 1 \rangle$, they are $(-1 + t, 2 - t, t)$, all nonnegative when $1 \leq t \leq 2$.

For $\langle 1, 1, -1 \rangle$ and $\langle 1, -1, -1 \rangle$, the requirements $x \geq 0$ and $z \geq 0$ force incompatible signs of t . Thus exactly the first and third listed lines enter the first octant.

Verification The stated intervals satisfy all three coordinate inequalities, including the boundary planes.

Question 8

Two lines intersect at $Q = (1, -2, 3)$ and have direction vectors

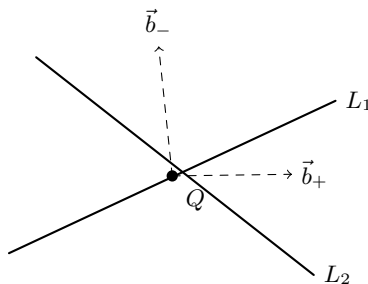
$$\vec{d}_1 = \langle 3, 0, 4 \rangle, \quad \vec{d}_2 = \langle 0, 4, 3 \rangle.$$

Tasks

1. Find equations of both angle-bisector lines through Q .
2. Determine which bisector corresponds to the acute angle between the given lines.
3. Prove that the two bisector lines are perpendicular.

Question 8 – Solution

Strategy Angle-bisector directions are the sum and difference of unit direction vectors. Here the given vectors already have equal length.



Bisector directions Since $\|\vec{d}_1\| = \|\vec{d}_2\| = 5$, the sum and difference give

$$\vec{b}_+ = \langle 3, 4, 7 \rangle, \quad \vec{b}_- = \langle 3, -4, 1 \rangle.$$

Therefore the bisectors are

$$\boxed{\vec{r} = \langle 1, -2, 3 \rangle + t \langle 3, 4, 7 \rangle}, \quad \boxed{\vec{r} = \langle 1, -2, 3 \rangle + u \langle 3, -4, 1 \rangle}.$$

Acute-angle bisector Because $\vec{d}_1 \cdot \vec{d}_2 = 12 > 0$, the directions form an acute angle. Their sum $\vec{b}_+$ lies between them and bisects that acute angle; $\vec{b}_-$ bisects the supplementary angle.

Perpendicularity Directly,

$$\vec{b}_+ \cdot \vec{b}_- = 3(3) + 4(-4) + 7(1) = 9 - 16 + 7 = 0.$$

Hence the two angle-bisector lines are perpendicular.

Verification Dotting $\vec{b}_+$ with each original direction gives the same normalized cosine, confirming equal angles.

Question 9

A line intersects the xy -plane at

$$A = (2, -1, 0)$$

and the yz -plane at

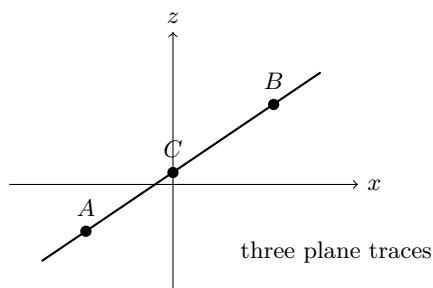
$$B = (0, 3, 4).$$

Tasks

1. Write vector, parametric, and symmetric equations of the line.
2. Find its intersection with the xz -plane.
3. Determine the ratio in which the xz -plane intersection divides $\overline{AB}$.

Question 9 – Solution

Strategy The two traces determine the line. Parameterize from A toward B , then impose $y = 0$ for the xz -plane.



Line equations A direction is $B - A = \langle -2, 4, 4 \rangle = 2 \langle -1, 2, 2 \rangle$. Thus

$$\vec{r} = \langle 2, -1, 0 \rangle + t \langle -1, 2, 2 \rangle,$$

$$x = 2 - t, \quad y = -1 + 2t, \quad z = 2t,$$

and

$$\frac{x - 2}{-1} = \frac{y + 1}{2} = \frac{z}{2}.$$

xz -plane trace Set $y = 0$. Then $-1 + 2t = 0$, so $t = 1/2$ and

$$C = \left(\frac{3}{2}, 0, 1 \right).$$

Division ratio In this reduced parameterization, B occurs at $t = 2$. Thus C lies one-half unit from A in parameter and three-halves units from B , giving

$$AC : CB = 1 : 3.$$

Verification C has $y = 0$, and substituting its coordinates into the symmetric equation gives the common value $1/2$.

Question 10

Two particles move for $t \geq 0$ according to

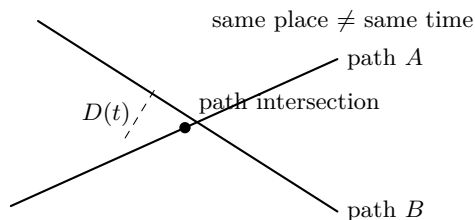
$$\begin{aligned}\vec{r}_A(t) &= \langle 0, 1, 2 \rangle + t \langle 2, -1, 1 \rangle, \\ \vec{r}_B(t) &= \langle 6, -2, 5 \rangle + t \langle -1, 1, 0 \rangle.\end{aligned}$$

Tasks

1. Determine whether their geometric paths intersect.
2. Determine whether the particles collide.
3. Find the time and value of their minimum separation for $t \geq 0$.

Question 10 – Solution

Strategy Path intersection allows different parameters, whereas collision uses the same time. Minimize the squared norm of the relative-position vector.



Paths versus collision Solving $\vec{r}_A(s) = \vec{r}_B(u)$ gives $s = 3$, $u = 0$, so the paths intersect at $\boxed{(6, -2, 5)}$. The particles reach that point at different times and therefore do not collide. Equating positions at the same t would require all components of the relative vector below to vanish, which never occurs.

Separation At a common time t ,

$$\vec{r}_A(t) - \vec{r}_B(t) = \langle -6 + 3t, 3 - 2t, -3 + t \rangle.$$

Hence

$$D^2(t) = (-6 + 3t)^2 + (3 - 2t)^2 + (-3 + t)^2 = 14t^2 - 54t + 54.$$

The vertex occurs at

$$t = \frac{54}{28} = \frac{27}{14} > 0.$$

Substitution gives $D_{\min}^2 = 27/14$, so

$$\boxed{D_{\min} = \sqrt{\frac{27}{14}} = \frac{3\sqrt{42}}{14}}.$$

Verification The quadratic opens upward, and the minimizing time lies in the allowed domain. The nonzero minimum confirms independently that no collision occurs.