

Question 1

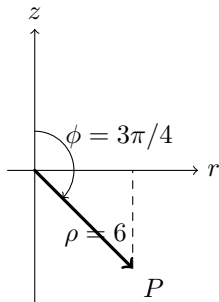
Convert the Cartesian point $P = (-3, 3, -3\sqrt{2})$ to spherical coordinates using $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

Tasks

1. Find ρ , θ , and ϕ .
2. Give one equivalent spherical representation outside the principal angular range.
3. Verify by converting back.

Question 1 – Solution

Strategy. Find radial distance first, use the quadrant of (x, y) for θ , and use $\cos \phi = z/\rho$.



Step 1: Radius

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{9 + 9 + 18} = 6.$$

Step 2: Angles The projection $(-3, 3)$ lies in quadrant II, so $\theta = 3\pi/4$. Also

$$\cos \phi = \frac{-3\sqrt{2}}{6} = -\frac{\sqrt{2}}{2}, \quad 0 \leq \phi \leq \pi,$$

giving $\phi = 3\pi/4$. Hence

$$\boxed{(\rho, \theta, \phi) = (6, 3\pi/4, 3\pi/4)}.$$

Step 3: Equivalence and verification Adding 2π to the azimuth gives $(6, 11\pi/4, 3\pi/4)$. Moreover,

$$6 \sin(3\pi/4) \cos(3\pi/4) = -3, \quad 6 \sin(3\pi/4) \sin(3\pi/4) = 3, \quad 6 \cos(3\pi/4) = -3\sqrt{2}.$$

Question 2

Convert $(\rho, \theta, \phi) = (8, 5\pi/6, \pi/3)$ to Cartesian and cylindrical coordinates.

Tasks

1. Find (x, y, z) exactly.
2. Find the corresponding principal cylindrical coordinates.
3. Verify both radial identities.

Question 2 – Solution

Strategy. Use $r = \rho \sin \phi$ and $z = \rho \cos \phi$, then resolve r using θ .

Step 1: Cylindrical data

$$r = 8 \sin(\pi/3) = 4\sqrt{3}, \quad z = 8 \cos(\pi/3) = 4.$$

Thus the principal cylindrical coordinates are

$$(r, \theta, z) = (4\sqrt{3}, 5\pi/6, 4).$$

Step 2: Cartesian data

$$x = r \cos \theta = (4\sqrt{3})(-\sqrt{3}/2) = -6, \quad y = r \sin \theta = (4\sqrt{3})(1/2) = 2\sqrt{3}.$$

Therefore

$$(x, y, z) = (-6, 2\sqrt{3}, 4).$$

Verification. $x^2 + y^2 = 36 + 12 = 48 = r^2$, and $x^2 + y^2 + z^2 = 48 + 16 = 64 = \rho^2$.

Question 3

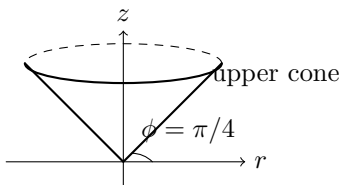
Describe geometrically the coordinate surfaces $\rho = 5$, $\theta = 2\pi/3$, and $\phi = \pi/4$.

Tasks

1. Identify each full coordinate surface.
2. Describe the pairwise intersections.
3. Find their common point.

Question 3 – Solution

Strategy. Interpret ρ as distance, θ as azimuth, and ϕ as inclination from the positive z -axis.



Step 1: Surfaces $\rho = 5$ is the sphere of radius 5. The equation $\theta = 2\pi/3$ is a vertical half-plane from the z -axis in azimuth $2\pi/3$. The equation $\phi = \pi/4$ is the upper nappe of the cone making angle $\pi/4$ with the positive z -axis.

Step 2: Pairwise intersections Sphere–half-plane and sphere–cone intersections are semicircle and horizontal circle, respectively. Half-plane–cone is a ray from the origin.

Step 3: Common point Substitution gives

$$x = 5\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) = -\frac{5\sqrt{2}}{4}, \quad y = 5\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{6}}{4}, \quad z = \frac{5\sqrt{2}}{2}.$$

Thus

$$P = \left(-5\sqrt{2}/4, 5\sqrt{6}/4, 5\sqrt{2}/2\right).$$

Question 4

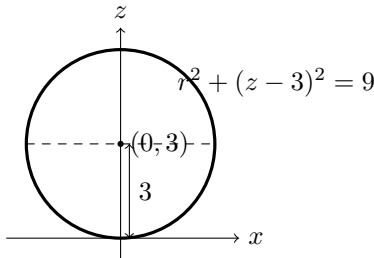
Rewrite the sphere $x^2 + y^2 + z^2 = 6z$ in spherical coordinates and identify its geometry.

Tasks

1. Obtain the nontrivial spherical equation.
2. Find its Cartesian center and radius.
3. Explain the role of $\rho = 0$ when simplifying.

Question 4 – Solution

Strategy. Substitute $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho \cos \phi$, while preserving any solution lost by division.

**Step 1: Convert**

$$\rho^2 = 6\rho \cos \phi \iff \rho(\rho - 6 \cos \phi) = 0.$$

The usual surface equation is

$$\boxed{\rho = 6 \cos \phi}, \quad 0 \leq \phi \leq \frac{\pi}{2},$$

and it includes the origin at $\phi = \pi/2$.

Step 2: Identify Completing the square in Cartesian form gives

$$x^2 + y^2 + (z - 3)^2 = 9.$$

It is the sphere centered at $\boxed{(0, 0, 3)}$ with radius $\boxed{3}$.

Step 3: Origin check Division by ρ would temporarily discard $\rho = 0$. The origin satisfies the original equation and is recovered by $\rho = 6 \cos(\pi/2) = 0$; thus no point is absent from the final parametrized surface.

Question 5

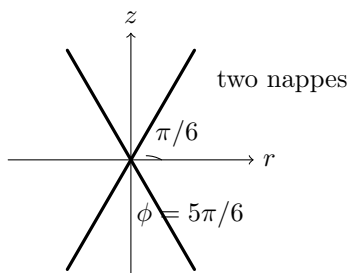
Convert the surface $z^2 = 3(x^2 + y^2)$ to spherical coordinates and determine exactly which cone it represents.

Tasks

1. Derive all permitted constant values of ϕ .
2. Identify both nappes.
3. State the status of the origin.

Question 5 – Solution

Strategy. Substitute $z = \rho \cos \phi$ and $x^2 + y^2 = \rho^2 \sin^2 \phi$, treating $\rho = 0$ separately.



Step 1: Convert For $\rho > 0$,

$$\rho^2 \cos^2 \phi = 3\rho^2 \sin^2 \phi \implies \tan^2 \phi = \frac{1}{3}.$$

On $0 \leq \phi \leq \pi$, the solutions are

$$\boxed{\phi = \frac{\pi}{6} \quad \text{or} \quad \phi = \frac{5\pi}{6}}.$$

Step 2: Geometry The first value is the upper nappe, $z \geq 0$; the second is the lower nappe, $z \leq 0$. Each makes an acute angle $\pi/6$ with its adjacent half of the z -axis.

Step 3: Origin At $\rho = 0$, every angle represents the origin, which satisfies the Cartesian equation. Therefore the two nappes meet at the included vertex $\boxed{(0, 0, 0)}$.

Question 6

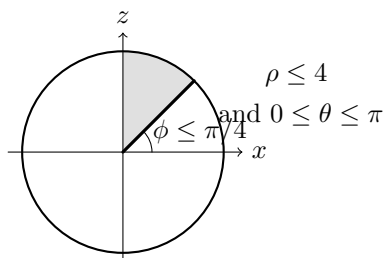
Describe in spherical inequalities the solid inside $x^2 + y^2 + z^2 \leq 16$, above the cone $z = \sqrt{x^2 + y^2}$, and in the half-space $y \geq 0$.

Tasks

1. Determine bounds for ρ , θ , and ϕ .
2. Explain each angular bound geometrically.
3. State which boundary pieces meet at the origin.

Question 6 – Solution

Strategy. Translate the sphere to a radial bound, the half-space to azimuth, and compare z with the cylindrical radius to bound inclination.



Step 1: Radial bound The sphere gives $0 \leq \rho \leq 4$.

Step 2: Azimuth Since $y = \rho \sin \phi \sin \theta$ and $\sin \phi \geq 0$, $y \geq 0$ is represented without overlap by $0 \leq \theta \leq \pi$.

Step 3: Inclination Above the cone means

$$\rho \cos \phi \geq \rho \sin \phi.$$

For $\rho > 0$ this gives $0 \leq \phi \leq \pi/4$. Consequently

$$\boxed{0 \leq \rho \leq 4, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq \pi/4}.$$

The sphere is the outer boundary; the cone and the two half-plane faces $\theta = 0, \pi$ all meet at the origin. Angular multiplicity there does not duplicate physical volume.

Question 7

Find the distance between $P = (3, \pi/6, \pi/3)$ and $Q = (5, 5\pi/6, 2\pi/3)$ in spherical coordinates without first converting both points to Cartesian form.

Tasks

1. Derive the angle between their radial directions.
2. Compute the distance exactly.
3. Check that the result obeys the triangle bounds.

Question 7 – Solution

Strategy. Take the dot product of the two unit radial directions and apply the three-dimensional law of cosines.

Step 1: Direction angle If γ is the angle between the radial directions, then

$$\cos \gamma = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \cos(\theta_1 - \theta_2).$$

Here $\cos \phi_1 \cos \phi_2 = -1/4$, $\sin \phi_1 \sin \phi_2 = 3/4$, and $\cos(-2\pi/3) = -1/2$, so $\cos \gamma = -5/8$.

Step 2: Distance Therefore

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos \gamma = 9 + 25 - 30(-5/8) = \frac{211}{4}.$$

Thus

$$d = \frac{\sqrt{211}}{2}.$$

Verification. The triangle inequality requires $|5-3| \leq d \leq 5+3$. Numerically $d \approx 7.26$, which lies between 2 and 8.

Question 8

Analyze all spherical-coordinate representations of points on the z -axis under the conventions $\rho \geq 0$, $0 \leq \theta < 2\pi$, and $0 \leq \phi \leq \pi$.

Tasks

1. Characterize a point $(0, 0, a)$ for $a > 0$, $a < 0$, and $a = 0$.
2. Identify which coordinates are nonunique.
3. Explain why the conversion formulas remain consistent.

Question 8 – Solution

Strategy. On the axis, the horizontal projection has length $\rho \sin \phi = 0$, so analyze the poles and origin separately.

Step 1: Positive axis If $a > 0$, then $\rho = a$, $\phi = 0$, and θ is arbitrary:

$$\boxed{(a, \theta, 0), \quad 0 \leq \theta < 2\pi}.$$

Step 2: Negative axis If $a < 0$, then $\rho = |a|$, $\phi = \pi$, and again θ is arbitrary:

$$\boxed{(|a|, \theta, \pi), \quad 0 \leq \theta < 2\pi}.$$

At the origin, $\rho = 0$ and both angles are arbitrary in their stated ranges.

Step 3: Consistency At $\phi = 0$ or π , $\sin \phi = 0$, so $x = y = 0$ regardless of θ , while $z = \rho \cos \phi$ has the correct sign. At $\rho = 0$, all three Cartesian coordinates vanish regardless of either angle. Thus the nonuniqueness is coordinate degeneracy, not geometric ambiguity.

Question 9

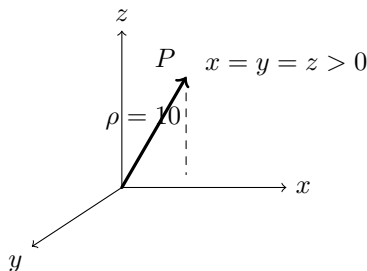
A point P has spherical coordinates (ρ, θ, ϕ) with $\rho = 10$ and is equally distant from the positive x -, y -, and z -axes. Assume P lies in the first octant.

Tasks

1. Determine the Cartesian coordinates of P .
2. Find its principal spherical angles.
3. Verify the three axis distances directly.

Question 9 – Solution

Strategy. Express squared distance to each coordinate axis and use first-octant signs.



Step 1: Equal-distance conditions Squared distances to the x -, y -, and z -axes are $y^2 + z^2$, $x^2 + z^2$, and $x^2 + y^2$. Equality implies $x^2 = y^2 = z^2$. In the first octant, $x = y = z$.

Step 2: Radius condition Since $x^2 + y^2 + z^2 = 100$, we obtain $3x^2 = 100$ and

$$P = (10/\sqrt{3}, 10/\sqrt{3}, 10/\sqrt{3}).$$

Then $\theta = \pi/4$, and $\cos \phi = z/\rho = 1/\sqrt{3}$, so

$$(\rho, \theta, \phi) = (10, \pi/4, \arccos(1/\sqrt{3})).$$

Verification. Every squared axis distance is $100/3 + 100/3 = 200/3$, so every distance is $10\sqrt{2/3}$.

Question 10

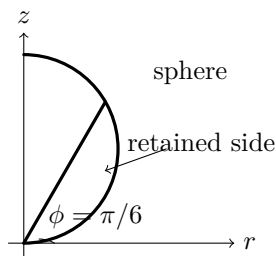
A solid is proposed in spherical coordinates by $0 \leq \rho \leq 6 \cos \phi$, $\pi/6 \leq \phi \leq 2\pi/3$, and $0 \leq \theta < 2\pi$.

Tasks

1. Determine where the radial bounds are consistent.
2. Correct the angular interval while retaining all actual points described.
3. Identify the solid and describe its boundary circle.

Question 10 – Solution

Strategy. A nonnegative radial variable can lie below $6 \cos \phi$ only when that upper bound is nonnegative.



Step 1: Consistency Since $\rho \geq 0$, we require $6 \cos \phi \geq 0$. On $0 \leq \phi \leq \pi$, this means $0 \leq \phi \leq \pi/2$. Intersecting with the proposed range gives

$$\boxed{\pi/6 \leq \phi \leq \pi/2}.$$

Angles $\pi/2 < \phi \leq 2\pi/3$ contribute no points because their proposed upper radial bound is negative.

Step 2: Correct description

$$\boxed{0 \leq \theta < 2\pi, \quad \pi/6 \leq \phi \leq \pi/2, \quad 0 \leq \rho \leq 6 \cos \phi}.$$

The radial surface is the sphere $x^2 + y^2 + (z - 3)^2 = 9$. The lower angular cutoff removes the portion closer than $\pi/6$ to the positive z -axis, leaving the part of the ball outside that narrow cone.

Step 3: Boundary circle On $\phi = \pi/6$ and the sphere, $\rho = 3\sqrt{3}$. Hence

$$z = \rho \cos \phi = \frac{9}{2}, \quad r = \rho \sin \phi = \frac{3\sqrt{3}}{2}.$$

The surfaces meet in

$$\boxed{x^2 + y^2 = 27/4, \quad z = 9/2}.$$