

Question 1

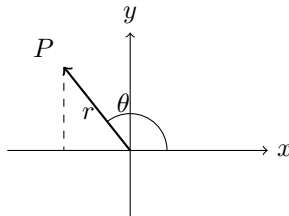
Convert $P = (-3, 3\sqrt{3}, 4)$ from Cartesian to cylindrical coordinates.

Tasks

1. Find r , a principal angle $0 \leq \theta < 2\pi$, and z .
2. Give two other equivalent cylindrical representations.
3. Verify by converting back.

Question 1 – Solution

Strategy. Use $r = \sqrt{x^2 + y^2}$ and determine the quadrant before choosing θ .



Step 1: Radius $r = \sqrt{9 + 27} = 6$.

Step 2: Angle Since $x < 0$ and $y > 0$, the point is in quadrant II. Also $\tan \theta = -\sqrt{3}$, so $\theta = 2\pi/3$. The height is unchanged. Thus

$$(r, \theta, z) = (6, 2\pi/3, 4).$$

Step 3: Equivalent forms Adding 2π gives $(6, 8\pi/3, 4)$. Reversing the radial sign and adding π gives $(-6, 5\pi/3, 4)$.

Verification. $6 \cos(2\pi/3) = -3$, $6 \sin(2\pi/3) = 3\sqrt{3}$, and $z = 4$.

Question 2

Convert $(r, \theta, z) = (4, 7\pi/6, -2)$ to Cartesian coordinates and determine its distance from the z -axis and from the origin.

Tasks

1. Find (x, y, z) .
2. Find both distances.
3. Verify the radial identity.

Question 2 – Solution

Strategy. Use $x = r \cos \theta$, $y = r \sin \theta$; radial distance is r when $r \geq 0$.

Step 1: Convert Since $\cos(7\pi/6) = -\sqrt{3}/2$ and $\sin(7\pi/6) = -1/2$,

$$\boxed{(x, y, z) = (-2\sqrt{3}, -2, -2)}.$$

Step 2: Distances Distance to the z -axis is $\sqrt{x^2 + y^2} = 4$. Distance to the origin is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{16 + 4} = \boxed{2\sqrt{5}}.$$

Verification. $x^2 + y^2 = 12 + 4 = 16 = r^2$, confirming the conversion and axis distance.

Question 3

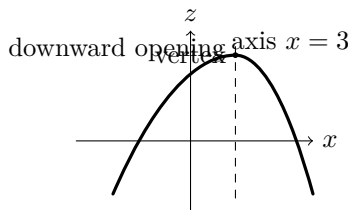
Rewrite $x^2 + y^2 - 6x + 4z = 3$ in cylindrical coordinates and identify the surface.

Tasks

1. Convert and solve for z .
2. Describe horizontal traces.
3. Locate the vertex and axis.

Question 3 – Solution

Strategy. Replace $x^2 + y^2$ by r^2 and x by $r \cos \theta$.



Step 1: Convert

$$r^2 - 6r \cos \theta + 4z = 3, \quad \boxed{z = \frac{3 - r^2 + 6r \cos \theta}{4}}.$$

For geometric identification, complete the Cartesian square:

$$(x - 3)^2 + y^2 = 12 - 4z.$$

Step 2: Identify Equivalently,

$$z = 3 - \frac{(x - 3)^2 + y^2}{4}.$$

This is a downward-opening circular paraboloid with vertical axis $x = 3, y = 0$ and vertex $\boxed{(3, 0, 3)}$.

Step 3: Traces At height $z = c$, the trace is a circle centered at $(3, 0, c)$ with radius $2\sqrt{3 - c}$, present for $c \leq 3$.

Question 4

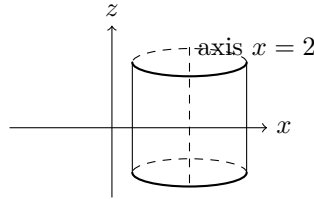
Describe the surface $r = 4 \cos \theta$ in Cartesian coordinates.

Tasks

1. Convert without dividing by r .
2. Identify the surface and its axis.
3. State its intersections with the coordinate axes.

Question 4 – Solution

Strategy. Multiply by r and use $r^2 = x^2 + y^2$ and $r \cos \theta = x$.



Step 1: Convert From $r = 4 \cos \theta$,

$$r^2 = 4r \cos \theta \implies x^2 + y^2 = 4x.$$

Completing the square,

$$\boxed{(x - 2)^2 + y^2 = 4}.$$

Step 2: Identify Since z is unrestricted, this is a circular cylinder of radius 2 whose axis is the vertical line $x = 2, y = 0$.

Step 3: Axis intersections In the xy -plane it meets the x -axis at $(0, 0, 0)$ and $(4, 0, 0)$ and the y -axis only at the origin. More generally, the cylinder contains the vertical lines $(0, 0, z)$ and $(4, 0, z)$.

Question 5

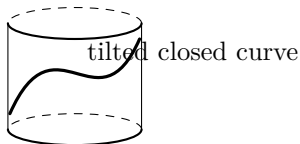
Find a cylindrical parametrization of the curve where $x^2 + y^2 = 9$ and $z = x + y$.

Tasks

1. Express both surfaces cylindrically.
2. Parametrize the entire intersection once.
3. Find its highest and lowest points.

Question 5 – Solution

Strategy. The cylinder fixes r ; let θ be the curve parameter and use the plane for z .



Step 1: Constraints The cylinder gives $r = 3$. The plane becomes

$$z = r \cos \theta + r \sin \theta = 3(\cos \theta + \sin \theta).$$

Thus

$$\mathbf{r}(\theta) = \langle 3 \cos \theta, 3 \sin \theta, 3(\cos \theta + \sin \theta) \rangle, \quad 0 \leq \theta < 2\pi.$$

Step 2: Extrema Use $\cos \theta + \sin \theta = \sqrt{2} \cos(\theta - \pi/4)$. Hence $z_{\max} = 3\sqrt{2}$ at $\theta = \pi/4$, and $z_{\min} = -3\sqrt{2}$ at $\theta = 5\pi/4$.

Step 3: Points

$$P_{\max} = (3/\sqrt{2}, 3/\sqrt{2}, 3\sqrt{2}), \quad P_{\min} = (-3/\sqrt{2}, -3/\sqrt{2}, -3\sqrt{2}).$$

Both satisfy the original cylinder and plane equations.

Question 6

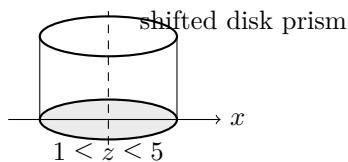
Convert the region $x^2 + y^2 \leq 4x$, $1 \leq z \leq 5$ to cylindrical inequalities.

Tasks

1. Determine the correct angular interval.
2. Give radial and vertical bounds.
3. Explain the geometry and why negative radial bounds are unnecessary.

Question 6 – Solution

Strategy. Convert the disk inequality and enforce the convention $r \geq 0$.



Step 1: Convert The inequality becomes

$$r^2 \leq 4r \cos \theta.$$

With $r \geq 0$, this is $0 \leq r \leq 4 \cos \theta$. The upper bound must be nonnegative, so $\cos \theta \geq 0$.

Step 2: Bounds One nonoverlapping description is

$$\boxed{-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 4 \cos \theta, \quad 1 \leq z \leq 5}.$$

Step 3: Geometry Completing the square gives $(x - 2)^2 + y^2 \leq 4$: a radius-2 disk centered at $(2, 0)$, extended from height 1 to 5. Negative r is unnecessary because directions with $\cos \theta < 0$ would duplicate points already represented using nonnegative r and opposite angles.

Question 7

Find the distance between cylindrical points $P = (2, \pi/6, 1)$ and $Q = (5, 5\pi/6, 4)$ without first listing both Cartesian points.

Tasks

1. Derive a cylindrical distance formula.
2. Evaluate it exactly.
3. Check the horizontal contribution geometrically.

Question 7 – Solution

Strategy. Expand $(x_1 - x_2)^2 + (y_1 - y_2)^2$ and use the cosine difference identity.

Step 1: Formula For cylindrical points,

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2.$$

This follows from the planar law of cosines plus the vertical difference.

Step 2: Substitute Here $\theta_1 - \theta_2 = -2\pi/3$, so its cosine is $-1/2$. Thus

$$\begin{aligned} d^2 &= 2^2 + 5^2 - 2(2)(5)(-1/2) + (1 - 4)^2 \\ &= 4 + 25 + 10 + 9 = 48. \end{aligned}$$

Therefore $\boxed{d = 4\sqrt{3}}$.

Step 3: Check The horizontal squared distance is $4 + 25 + 10 = 39$; adding vertical squared distance 9 gives 48, consistent with perpendicular horizontal and vertical components.

Question 8

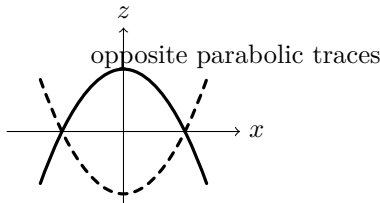
Analyze the surface $z = r^2 \cos 2\theta$.

Tasks

1. Convert it to Cartesian form.
2. Identify its principal vertical traces.
3. Describe its signs by angular sector and its symmetry.

Question 8 – Solution

Strategy. Use $r^2 \cos 2\theta = x^2 - y^2$.



Step 1: Convert Since $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$,

$$r^2 \cos 2\theta = r^2 \cos^2 \theta - r^2 \sin^2 \theta = x^2 - y^2.$$

Thus $\boxed{z = x^2 - y^2}$, a hyperbolic paraboloid.

Step 2: Traces In $y = 0$, $z = x^2$ opens upward. In $x = 0$, $z = -y^2$ opens downward. On $y = \pm x$, $z = 0$.

Step 3: Angular behavior The height is positive where $\cos 2\theta > 0$, negative where it is negative, and zero at $\theta = \pi/4 + k\pi/2$. The equation is unchanged by $\theta \mapsto \theta + \pi$ and by reflections across either coordinate axis, matching the Cartesian symmetries.

Question 9

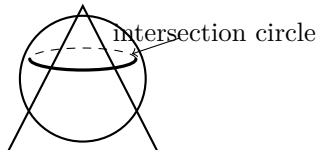
A point has cylindrical coordinates (r, θ, z) with $r = 2z$, $z \geq 0$, and lies on the sphere $x^2 + y^2 + z^2 = 45$.

Tasks

1. Determine r and z .
2. Describe all possible points.
3. Find the horizontal circle containing them.

Question 9 – Solution

Strategy. Convert the sphere to $r^2 + z^2 = 45$ and combine it with the cone relation.



Step 1: Solve Substitute $r = 2z$:

$$(2z)^2 + z^2 = 45 \implies 5z^2 = 45.$$

Since $z \geq 0$, $\boxed{z = 3}$ and $\boxed{r = 6}$.

Step 2: Angular freedom No condition restricts θ , so

$$\boxed{(r, \theta, z) = (6, \theta, 3), \quad 0 \leq \theta < 2\pi}.$$

In Cartesian form the points are $(6 \cos \theta, 6 \sin \theta, 3)$.

Step 3: Locus They form the horizontal circle

$$\boxed{x^2 + y^2 = 36, \quad z = 3}.$$

Verification gives $36 + 9 = 45$ and $r = 6 = 2(3)$.

Question 10

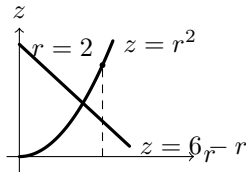
A solid is described cylindrically by $0 \leq r \leq 2 + \cos \theta$, $0 \leq \theta < 2\pi$, and $r^2 \leq z \leq 6 - r$.

Tasks

1. Determine whether the vertical bounds are consistent everywhere in the stated radial range.
2. Correct the radial upper bound if necessary.
3. Give a precise final description and explain which constraint is active.

Question 10 – Solution

Strategy. Vertical consistency requires lower height not exceed upper height.

**Step 1: Compare heights**

$$r^2 \leq 6 - r \iff r^2 + r - 6 \leq 0 \iff (r + 3)(r - 2) \leq 0.$$

With $r \geq 0$, this requires $0 \leq r \leq 2$.

Step 2: Combine radial restrictions The stated planar bound is $r \leq 2 + \cos \theta$, which ranges from 1 to 3. Both conditions must hold, so

$$0 \leq r \leq \min\{2, 2 + \cos \theta\}.$$

Equivalently, the upper bound is 2 when $\cos \theta \geq 0$ and $2 + \cos \theta$ when $\cos \theta < 0$.

Step 3: Final description

$$0 \leq \theta < 2\pi, \quad 0 \leq r \leq \min\{2, 2 + \cos \theta\}, \quad r^2 \leq z \leq 6 - r.$$

The height surfaces meet at $r = 2$. For angles on the right half-plane the height condition truncates the planar radius; on the left half-plane the limaçon-type boundary is already smaller than 2.