

Question 1

A particle has position

$$\mathbf{r}(t) = \langle t^2 - 2t, t^3 - 3t, 4t + 1 \rangle, \quad t \geq 0.$$

Tasks

1. Find velocity, acceleration, and speed.
2. Evaluate them at $t = 1$.
3. Find the tangent line to the trajectory at that time.

Question 1 – Solution

Strategy. Differentiate position componentwise; speed is the magnitude of velocity.

Step 1: Velocity and acceleration

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t - 2, 3t^2 - 3, 4 \rangle,$$

and

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 0 \rangle.$$

Therefore

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{(2t - 2)^2 + (3t^2 - 3)^2 + 16}.$$

Step 2: Values at $t = 1$

$$\mathbf{r}(1) = \langle -1, -2, 5 \rangle, \quad \mathbf{v}(1) = \langle 0, 0, 4 \rangle, \quad \mathbf{a}(1) = \langle 2, 6, 0 \rangle.$$

The speed is $\boxed{4}$.

Step 3: Tangent line Since $\mathbf{v}(1) \neq \mathbf{0}$, it gives the tangent direction:

$$\mathbf{L}(s) = \langle -1, -2, 5 \rangle + s \langle 0, 0, 1 \rangle.$$

The simplified direction is parallel to velocity. At $s = 0$ the line passes through the correct position.

Question 2

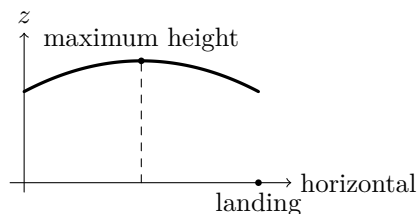
A ball is launched from $(0, 0, 5)$ with initial velocity $\langle 30, 40, 50 \rangle$ ft/s. Gravity is $\langle 0, 0, -32 \rangle$ ft/s² and air resistance is neglected.

Tasks

1. Find position and velocity for $t \geq 0$.
2. Find the maximum height and when it occurs.
3. Find the landing time and horizontal landing point.

Question 2 – Solution

Strategy. Integrate constant acceleration twice and apply the initial data; landing occurs when the vertical coordinate is zero.

**Step 1: Motion functions**

$$\mathbf{v}(t) = \langle 30, 40, 50 - 32t \rangle.$$

Integrating again and using $\mathbf{r}(0) = \langle 0, 0, 5 \rangle$,

$$\mathbf{r}(t) = \langle 30t, 40t, 5 + 50t - 16t^2 \rangle.$$

Step 2: Maximum height The vertical velocity is zero when

$$50 - 32t = 0 \implies t = \frac{25}{16}.$$

Then

$$z = 5 + 50\left(\frac{25}{16}\right) - 16\left(\frac{25}{16}\right)^2 = \boxed{\frac{705}{16} \text{ ft}}.$$

Step 3: Landing Set $z = 0$:

$$5 + 50t - 16t^2 = 0 \implies t = \frac{25 \pm \sqrt{705}}{16}.$$

Only the positive root is physical:

$$t_L = \frac{25 + \sqrt{705}}{16}.$$

Thus the horizontal landing point is

$$(x, y) = (30t_L, 40t_L).$$

Substitution into the vertical equation verifies height zero.

Question 3

A particle satisfies

$$\mathbf{a}(t) = \langle 6t, 4e^{2t}, -2 \rangle, \quad \mathbf{v}(0) = \langle 1, -1, 3 \rangle, \quad \mathbf{r}(0) = \langle 2, 0, -4 \rangle.$$

Tasks

1. Recover $\mathbf{v}(t)$.
2. Recover $\mathbf{r}(t)$.
3. Verify both initial conditions and the acceleration.

Question 3 – Solution

Strategy. Integrate componentwise twice, determining a new constant vector at each stage.

Step 1: Velocity

$$\mathbf{v}(t) = \langle 3t^2 + C_1, 2e^{2t} + C_2, -2t + C_3 \rangle.$$

At $t = 0$, comparison with $\langle 1, -1, 3 \rangle$ gives

$$C_1 = 1, \quad 2 + C_2 = -1, \quad C_3 = 3.$$

Therefore

$$\boxed{\mathbf{v}(t) = \langle 3t^2 + 1, 2e^{2t} - 3, 3 - 2t \rangle}.$$

Step 2: Position Integrating velocity,

$$\mathbf{r}(t) = \langle t^3 + t + D_1, e^{2t} - 3t + D_2, 3t - t^2 + D_3 \rangle.$$

Using $\mathbf{r}(0) = \langle 2, 0, -4 \rangle$ gives $D_1 = 2$, $1 + D_2 = 0$, and $D_3 = -4$. Hence

$$\boxed{\mathbf{r}(t) = \langle t^3 + t + 2, e^{2t} - 3t - 1, 3t - t^2 - 4 \rangle}.$$

Step 3: Verification Differentiating $\mathbf{r}$ gives the displayed $\mathbf{v}$; differentiating again gives $\langle 6t, 4e^{2t}, -2 \rangle$. Direct substitution at $t = 0$ returns both required initial vectors.

Question 4

A particle moves on a circle according to

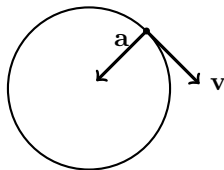
$$\mathbf{r}(t) = \langle R \cos(\omega t), R \sin(\omega t), 0 \rangle, \quad R > 0, \omega > 0.$$

Tasks

1. Find velocity, speed, and acceleration.
2. Prove velocity is perpendicular to acceleration.
3. Express acceleration in terms of position and identify its magnitude.

Question 4 – Solution

Strategy. Differentiate symbolically and use trigonometric identities and dot products.



velocity tangent; acceleration inward

Step 1: Differentiate

$$\mathbf{v} = \langle -R\omega \sin(\omega t), R\omega \cos(\omega t), 0 \rangle,$$

so

$$\|\mathbf{v}\| = R\omega.$$

Differentiating again,

$$\mathbf{a} = \langle -R\omega^2 \cos(\omega t), -R\omega^2 \sin(\omega t), 0 \rangle.$$

Step 2: Orthogonality

$$\mathbf{v} \cdot \mathbf{a} = R^2\omega^3 \sin(\omega t) \cos(\omega t) - R^2\omega^3 \cos(\omega t) \sin(\omega t) = 0.$$

Thus $\mathbf{v} \perp \mathbf{a}$.

Step 3: Inward acceleration Comparing with $\mathbf{r}$ gives

$$\mathbf{a}(t) = -\omega^2 \mathbf{r}(t).$$

It points toward the origin, and

$$\|\mathbf{a}\| = R\omega^2 = \frac{(R\omega)^2}{R} = \frac{v^2}{R}.$$

This is purely normal acceleration; the constant speed confirms there is no tangential component.

Question 5

For the helical motion

$$\mathbf{r}(t) = \langle 3 \cos 2t, 3 \sin 2t, 4t \rangle,$$

determine the velocity, acceleration, speed, and angle between velocity and the positive z -axis.

Tasks

1. Compute $\mathbf{v}$ and $\mathbf{a}$.
2. Show speed and the requested angle are constant.
3. Prove acceleration points horizontally toward the helix axis.

Question 5 – Solution

Strategy. Differentiate, then use the dot-product angle formula with $\mathbf{k} = \langle 0, 0, 1 \rangle$.

Step 1: Motion quantities

$$\mathbf{v}(t) = \langle -6 \sin 2t, 6 \cos 2t, 4 \rangle,$$

and

$$\mathbf{a}(t) = \langle -12 \cos 2t, -12 \sin 2t, 0 \rangle.$$

The speed is

$$\|\mathbf{v}\| = \sqrt{36(\sin^2 2t + \cos^2 2t) + 16} = 2\sqrt{13}.$$

Step 2: Direction angle If θ is the angle with the positive z -axis,

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{k}}{\|\mathbf{v}\| \|\mathbf{k}\|} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}.$$

Therefore

$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{13}} \right).$$

No t occurs, so both speed and angle are constant.

Step 3: Acceleration geometry Its horizontal component is

$$-4 \langle 3 \cos 2t, 3 \sin 2t, 0 \rangle,$$

which is opposite the radial position from the z -axis. Its z -component is zero, so acceleration points horizontally inward. Also $\mathbf{v} \cdot \mathbf{a} = 0$, consistent with constant speed.

Question 6

Two objects move according to

$$\mathbf{r}_1(t) = \langle t, 2t, 3 - t \rangle, \quad \mathbf{r}_2(t) = \langle 4 - t, t + 1, 2t - 2 \rangle, \quad t \geq 0.$$

Tasks

1. Determine whether they collide.
2. Find their velocities and relative velocity.
3. Verify your collision conclusion using relative position.

Question 6 – Solution

Strategy. A collision requires equal position vectors at the same time, not merely intersecting geometric paths.

Step 1: Match coordinates Equality requires

$$t = 4 - t, \quad 2t = t + 1, \quad 3 - t = 2t - 2.$$

The first gives $t = 2$, while the second gives $t = 1$. Since no single time satisfies even these two equations, the objects do not collide:

$$\boxed{\text{no collision for } t \geq 0}.$$

Step 2: Velocities

$$\mathbf{v}_1 = \langle 1, 2, -1 \rangle, \quad \mathbf{v}_2 = \langle -1, 1, 2 \rangle.$$

The velocity of object 1 relative to object 2 is

$$\boxed{\mathbf{v}_{1/2} = \mathbf{v}_1 - \mathbf{v}_2 = \langle 2, 1, -3 \rangle}.$$

Step 3: Relative-position check

$$\mathbf{r}_1 - \mathbf{r}_2 = \langle 2t - 4, t - 1, 5 - 3t \rangle.$$

For this vector to be zero, its first, second, and third components would require respectively $t = 2$, $t = 1$, and $t = 5/3$. The inconsistent times independently confirm that relative position never vanishes.

Question 7

A particle has velocity

$$\mathbf{v}(t) = \langle t - 2, 2t + 1, 2 \rangle, \quad t \geq 0.$$

Tasks

1. Find when its speed is smallest.
2. Find the minimum speed.
3. Show that velocity is perpendicular to acceleration at that instant and explain why.

Question 7 – Solution

Strategy. Minimize squared speed, which has the same minimizers as speed and avoids a square root.

Step 1: Squared speed

$$\begin{aligned}q(t) = \|\mathbf{v}(t)\|^2 &= (t-2)^2 + (2t+1)^2 + 4 \\&= 5t^2 + 9.\end{aligned}$$

On $t \geq 0$, this is minimized at $\boxed{t = 0}$.

Step 2: Minimum speed

$$\mathbf{v}(0) = \langle -2, 1, 2 \rangle, \quad \boxed{\|\mathbf{v}(0)\| = 3}.$$

Step 3: Orthogonality Acceleration is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 1, 2, 0 \rangle.$$

At $t = 0$,

$$\mathbf{v}(0) \cdot \mathbf{a}(0) = -2 + 2 + 0 = 0.$$

Also $q'(t) = 2\mathbf{v} \cdot \mathbf{a} = 10t$, so the vanishing dot product is exactly the zero derivative of squared speed. Here the minimum occurs at the endpoint, and the right-hand derivative is nonnegative immediately afterward.

Question 8

A particle starts at $P = (1, 0, 2)$ with velocity $\langle 2, -1, 0 \rangle$ and constant acceleration $\langle -1, 2, 3 \rangle$.

Tasks

1. Find position and velocity at time t .
2. Determine when the particle first reaches the plane $z = 8$.
3. Find its position and speed then.

Question 8 – Solution

Strategy. For constant acceleration, use $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}_0 + \frac{1}{2}t^2\mathbf{a}$.

Step 1: Motion functions

$$\mathbf{v}(t) = \langle 2 - t, -1 + 2t, 3t \rangle,$$

and

$$\mathbf{r}(t) = \left\langle 1 + 2t - \frac{1}{2}t^2, -t + t^2, 2 + \frac{3}{2}t^2 \right\rangle.$$

At $t = 0$, these reproduce both initial vectors.

Step 2: Reach $z = 8$

$$2 + \frac{3}{2}t^2 = 8 \quad \implies \quad t^2 = 4.$$

For forward time, $t = 2$.

Step 3: Position and speed

$$\mathbf{r}(2) = \langle 1 + 4 - 2, -2 + 4, 8 \rangle = \langle 3, 2, 8 \rangle.$$

Also

$$\mathbf{v}(2) = \langle 0, 3, 6 \rangle,$$

so

$$\text{speed} = \sqrt{0^2 + 3^2 + 6^2} = 3\sqrt{5}.$$

The vertical coordinate and positive time verify that this is the first forward-time crossing.

Question 9

A particle follows

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 4t \rangle, \quad 0 \leq t \leq 2\pi.$$

Tasks

1. Find its displacement and average velocity over the interval.
2. Find total distance traveled and average speed.
3. Explain why the horizontal motion contributes to distance but not displacement.

Question 9 – Solution

Strategy. Displacement uses endpoints; distance integrates speed. These measure different features of motion.

Step 1: Displacement

$$\mathbf{r}(0) = \langle 3, 0, 0 \rangle, \quad \mathbf{r}(2\pi) = \langle 3, 0, 8\pi \rangle.$$

Thus

$$\Delta \mathbf{r} = \langle 0, 0, 8\pi \rangle.$$

The elapsed time is 2π , so

$$\mathbf{v}_{\text{avg}} = \frac{\Delta \mathbf{r}}{2\pi} = \langle 0, 0, 4 \rangle.$$

Step 2: Distance and average speed

$$\mathbf{v}(t) = \langle -3 \sin t, 3 \cos t, 4 \rangle,$$

so $\|\mathbf{v}(t)\| = \sqrt{9 + 16} = 5$. Hence

$$D = \int_0^{2\pi} 5 \, dt = 10\pi,$$

and

$$\text{average speed} = \frac{10\pi}{2\pi} = 5.$$

Step 3: Interpretation During one revolution, the horizontal coordinates return to their starting values, so their net changes cancel in displacement. Nevertheless the particle continuously moves horizontally around the circle, adding positive length to the traveled path. Therefore distance 10π exceeds displacement magnitude 8π .

Question 10

Let a regular motion have velocity $\mathbf{v}$, speed $v = \|\mathbf{v}\|$, unit tangent $\mathbf{T}$, curvature κ , and principal normal $\mathbf{N}$.

Tasks

1. Derive $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$.
2. Derive formulas for the tangential and normal scalar components using only $\mathbf{v}$ and $\mathbf{a}$.
3. Explain the physical meaning of each component.

Question 10 – Solution

Strategy. Begin with $\mathbf{v} = v\mathbf{T}$, differentiate, and convert $d\mathbf{T}/dt$ through arc length.

Step 1: Differentiate velocity

$$\mathbf{a} = \frac{d}{dt}(v\mathbf{T}) = v'\mathbf{T} + v\frac{d\mathbf{T}}{dt}.$$

By the chain rule,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} = (\kappa\mathbf{N})v.$$

Therefore

$$\boxed{\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}}.$$

Step 2: Tangential component Dot with $\mathbf{T} = \mathbf{v}/v$:

$$a_T = \mathbf{a} \cdot \mathbf{T} = \boxed{\frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}}.$$

It also equals v' , the rate of change of speed.

Step 3: Normal component Orthogonality gives

$$\|\mathbf{a}\|^2 = a_T^2 + a_N^2.$$

Thus

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} = \boxed{\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}} = \kappa v^2.$$

The tangential component changes speed; the normal component changes direction. A motion can therefore accelerate even at constant speed if its path bends.