

Question 1

For the plane curve $\mathbf{r}(t) = \langle t, t^2, 0 \rangle$, find the curvature at $t = 1$.

Tasks

1. Compute $\mathbf{r}'$ and $\mathbf{r}''$.
2. Use $\kappa = \|\mathbf{r}' \times \mathbf{r}''\| / \|\mathbf{r}'\|^3$.
3. State the radius of curvature at the point.

Question 1 – Solution

Strategy. The cross-product formula avoids differentiating the unit tangent.

Step 1: Derivatives

$$\mathbf{r}'(t) = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}''(t) = \langle 0, 2, 0 \rangle.$$

At $t = 1$, $\mathbf{r}'(1) = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}'(1)\| = \sqrt{5}$.

Step 2: Cross product

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 1, 2, 0 \rangle \times \langle 0, 2, 0 \rangle = \langle 0, 0, 2 \rangle,$$

so its magnitude is 2. Therefore

$$\kappa(1) = \frac{2}{(\sqrt{5})^3} = \frac{2}{5\sqrt{5}}.$$

Step 3: Radius of curvature Since $\rho = 1/\kappa$,

$$\rho(1) = \frac{5\sqrt{5}}{2}.$$

Both values are positive, and their product is 1, verifying the reciprocal relationship.

Question 2

Find the curvature of the circular helix

$$\mathbf{r}(t) = \langle a \cos t, a \sin t, bt \rangle, \quad a > 0,$$

where b is constant.

Tasks

1. Derive $\kappa(t)$ symbolically.
2. Explain why it is constant.
3. Check the special cases $b = 0$ and increasing $|b|$.

Question 2 – Solution

Strategy. Compute the speed and cross product in symbolic form before simplifying.

Step 1: Derivatives and speed

$$\mathbf{r}' = \langle -a \sin t, a \cos t, b \rangle, \quad \mathbf{r}'' = \langle -a \cos t, -a \sin t, 0 \rangle,$$

and

$$\|\mathbf{r}'\| = \sqrt{a^2 + b^2}.$$

Step 2: Cross product

$$\mathbf{r}' \times \mathbf{r}'' = \langle ab \sin t, -ab \cos t, a^2 \rangle.$$

Its magnitude is

$$\sqrt{a^2 b^2 (\sin^2 t + \cos^2 t) + a^4} = a \sqrt{a^2 + b^2}.$$

Thus

$$\kappa(t) = \frac{a \sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

Step 3: Interpretation No t remains, so curvature is constant. When $b = 0$, the helix becomes a circle of radius a and $\kappa = 1/a$. As $|b|$ grows, $a/(a^2 + b^2)$ decreases: the helix stretches vertically and bends less sharply.

Question 3

For the graph $y = \ln x$, $x > 0$, determine where its curvature is greatest.

Tasks

1. Derive $\kappa(x)$ using the graph formula.
2. Maximize it exactly.
3. Give the point and maximum curvature.

Question 3 – Solution

Strategy. Use $\kappa = |y''|/[1 + (y')^2]^{3/2}$, then simplify before differentiating.

Step 1: Curvature function

$$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}.$$

Therefore

$$\begin{aligned}\kappa(x) &= \frac{1/x^2}{(1 + 1/x^2)^{3/2}} \\ &= \frac{x}{(x^2 + 1)^{3/2}}, \quad x > 0.\end{aligned}$$

Step 2: Critical point

$$\begin{aligned}\kappa'(x) &= (x^2 + 1)^{-3/2} - 3x^2(x^2 + 1)^{-5/2} \\ &= \frac{1 - 2x^2}{(x^2 + 1)^{5/2}}.\end{aligned}$$

Thus $\kappa'(x) = 0$ when $x = 1/\sqrt{2}$. The numerator is positive before this point and negative after it, so this is the unique maximum.

Step 3: Point and value The point is

$$\left(\frac{1}{\sqrt{2}}, -\frac{1}{2} \ln 2 \right).$$

At that value,

$$\kappa_{\max} = \frac{1/\sqrt{2}}{(3/2)^{3/2}} = \boxed{\frac{2}{3\sqrt{3}}}.$$

Also $\kappa \rightarrow 0$ as $x \rightarrow 0^+$ or $x \rightarrow \infty$, confirming the interior maximum.

Question 4

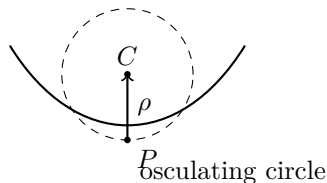
Find the osculating circle of $y = x^2$ at the vertex.

Tasks

1. Compute the curvature and principal-normal direction at $x = 0$.
2. Find the center and radius of the osculating circle.
3. Write its Cartesian equation and verify tangency.

Question 4 – Solution

Strategy. The center lies one radius of curvature from the point in the principal-normal direction.



Step 1: Curvature For $y = x^2$,

$$y' = 2x, \quad y'' = 2,$$

so

$$\kappa(0) = \frac{|2|}{(1+0)^{3/2}} = 2, \quad \rho(0) = \frac{1}{2}.$$

At the vertex the curve bends upward, so $\mathbf{N} = \langle 0, 1 \rangle$.

Step 2: Center With $P = (0, 0)$,

$$C = P + \rho \mathbf{N} = (0, 0) + \frac{1}{2}(0, 1) = \left(0, \frac{1}{2}\right).$$

Step 3: Circle equation Radius $1/2$ and center $(0, 1/2)$ give

$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

The circle contains $(0, 0)$. Implicit differentiation gives

$$2x + 2\left(y - \frac{1}{2}\right)y' = 0,$$

so at $(0, 0)$, $y' = 0$, matching the parabola's horizontal tangent. The common curvature at contact gives second-order agreement.

Question 5

The same parabola is parametrized by

$$\mathbf{r}(t) = \langle t, t^2, 0 \rangle \quad \text{and} \quad \mathbf{R}(u) = \langle u^3, u^6, 0 \rangle, \quad u > 0.$$

Tasks

1. Compute curvature from each parametrization.
2. Compare the answers at corresponding points $t = u^3$.
3. Explain why the restriction $u > 0$ is useful.

Question 5 – Solution

Strategy. Apply the cross-product formula twice and simplify all powers before comparing.

Step 1: First parametrization

$$\mathbf{r}' = \langle 1, 2t, 0 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 0 \rangle.$$

Thus

$$\kappa_r(t) = \frac{2}{(1 + 4t^2)^{3/2}}.$$

Step 2: Second parametrization

$$\mathbf{R}' = \langle 3u^2, 6u^5, 0 \rangle, \quad \mathbf{R}'' = \langle 6u, 30u^4, 0 \rangle.$$

The cross product has k -component

$$(3u^2)(30u^4) - (6u^5)(6u) = 54u^6.$$

Also

$$\|\mathbf{R}'\| = 3u^2\sqrt{1 + 4u^6}$$

because $u > 0$. Hence

$$\kappa_R(u) = \frac{54u^6}{27u^6(1 + 4u^6)^{3/2}} = \frac{2}{(1 + 4u^6)^{3/2}}.$$

Step 3: Compare Substituting $t = u^3$ into κ_r gives exactly κ_R . Curvature depends on the geometric curve, not its regular parametrization. The restriction $u > 0$ ensures $\mathbf{R}' \neq \mathbf{0}$ and selects the positive-parameter branch. The speed simplification uses $u^2 \geq 0$, which also holds for negative u .

Question 6

At a point of a regular curve,

$$\mathbf{r}' = \langle 2, -1, 2 \rangle, \quad \mathbf{r}'' = \langle 1, 2, 0 \rangle.$$

Tasks

1. Compute the curvature.
2. Decompose $\mathbf{r}''$ into components parallel and perpendicular to $\mathbf{r}'$.
3. Verify curvature using $\kappa = \|\mathbf{r}''_{\perp}\|/\|\mathbf{r}'\|^2$.

Question 6 – Solution

Strategy. Compare the cross-product formula with the geometric perpendicular-acceleration formula.

Step 1: Cross-product computation

$$\mathbf{r}' \times \mathbf{r}'' = \langle 2, -1, 2 \rangle \times \langle 1, 2, 0 \rangle = \langle -4, 2, 5 \rangle.$$

Its length is $\sqrt{16 + 4 + 25} = 3\sqrt{5}$, while $\|\mathbf{r}'\| = 3$. Therefore

$$\kappa = \frac{3\sqrt{5}}{3^3} = \frac{\sqrt{5}}{9}.$$

Step 2: Parallel component Since

$$\mathbf{r}'' \cdot \mathbf{r}' = 2 - 2 + 0 = 0,$$

the projection of $\mathbf{r}''$ onto $\mathbf{r}'$ is

$$\mathbf{r}''_{\parallel} = \frac{\mathbf{r}'' \cdot \mathbf{r}'}{\|\mathbf{r}'\|^2} \mathbf{r}' = \mathbf{0}.$$

Thus $\mathbf{r}''_{\perp} = \mathbf{r}'' = \langle 1, 2, 0 \rangle$ and $\|\mathbf{r}''_{\perp}\| = \sqrt{5}$.

Step 3: Verification

$$\frac{\|\mathbf{r}''_{\perp}\|}{\|\mathbf{r}'\|^2} = \frac{\sqrt{5}}{9},$$

which agrees exactly with the cross-product result.

Question 7

For the space curve

$$\mathbf{r}(t) = \langle t, t^2, t^3 \rangle,$$

find all real parameters at which the curvature is zero.

Tasks

1. Compute $\mathbf{r}' \times \mathbf{r}''$.
2. Decide whether its magnitude can vanish.
3. Interpret the conclusion geometrically.

Question 7 – Solution

Strategy. Since the curve is regular, curvature is zero exactly when $\mathbf{r}'$ and $\mathbf{r}''$ are parallel.

Step 1: Derivatives

$$\mathbf{r}' = \langle 1, 2t, 3t^2 \rangle, \quad \mathbf{r}'' = \langle 0, 2, 6t \rangle.$$

The first component of $\mathbf{r}'$ is always 1, so the curve is regular.

Step 2: Cross product

$$\begin{aligned} \mathbf{r}' \times \mathbf{r}'' &= \langle 1, 2t, 3t^2 \rangle \times \langle 0, 2, 6t \rangle \\ &= \langle 12t^2 - 6t^2, -6t, 2 \rangle \\ &= \langle 6t^2, -6t, 2 \rangle. \end{aligned}$$

Its squared magnitude is

$$36t^4 + 36t^2 + 4.$$

Every term is nonnegative and the constant term is $4 > 0$. Therefore the cross product never vanishes.

Step 3: Conclusion

$\kappa(t) > 0$ for every real t ; there are no zero-curvature points.

Geometrically, the tangent direction is changing everywhere. In particular, no point of this twisted cubic is locally straight in the curvature sense.

Question 8

Consider $\mathbf{r}(t) = \langle t, t^3, 0 \rangle$.

Tasks

1. Derive the curvature function.
2. Explain why curvature vanishes at the origin even though the curve is regular there.
3. Determine the two points where curvature is greatest.

Question 8 – Solution

Strategy. Use the planar graph formula and maximize the even function by working with $u = t^2$.

Step 1: Curvature

$$y' = 3t^2, \quad y'' = 6t,$$

so

$$\kappa(t) = \frac{6|t|}{(1 + 9t^4)^{3/2}}.$$

At $t = 0$, $\mathbf{r}'(0) = \langle 1, 0, 0 \rangle \neq \mathbf{0}$, so the curve is regular, but $y''(0) = 0$ and therefore $\kappa(0) = 0$. This is an inflection point.

Step 2: Maximize for $t > 0$ There $|t| = t$. Logarithmic differentiation gives

$$\frac{\kappa'}{\kappa} = \frac{1}{t} - \frac{54t^3}{1 + 9t^4}.$$

Setting this equal to zero,

$$1 + 9t^4 = 54t^4 \implies t^4 = \frac{1}{45}.$$

Thus $t = 45^{-1/4}$ on the positive side. By even symmetry, the other maximizer is its negative.

Step 3: Points With $\alpha = 45^{-1/4}$, the two curve points are

$$(\alpha, \alpha^3, 0) \quad \text{and} \quad (-\alpha, -\alpha^3, 0).$$

Curvature tends to zero as $|t| \rightarrow \infty$ and is zero at $t = 0$, so these critical points are global maxima.

Question 9

A helix must have radius 3 and constant curvature $1/5$. It is modeled by

$$\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, bt \rangle, \quad b > 0.$$

Tasks

1. Determine b exactly.
2. Find the vertical rise per complete revolution.
3. Verify the designed curvature directly.

Question 9 – Solution

Strategy. Use the helix curvature formula and then translate parameter change into vertical rise.

Step 1: Solve the design equation For a helix of radius a ,

$$\kappa = \frac{a}{a^2 + b^2}.$$

With $a = 3$ and $\kappa = 1/5$,

$$\begin{aligned}\frac{3}{9 + b^2} &= \frac{1}{5}, \\ 15 &= 9 + b^2, \\ b^2 &= 6.\end{aligned}$$

Since $b > 0$, $\boxed{b = \sqrt{6}}$.

Step 2: Rise per revolution A full revolution changes t by 2π . Since $z = bt$,

$$\Delta z = b(2\pi) = \boxed{2\pi\sqrt{6}}.$$

Step 3: Verification Substituting $b^2 = 6$ into the curvature formula gives

$$\kappa = \frac{3}{3^2 + (\sqrt{6})^2} = \frac{3}{9 + 6} = \boxed{\frac{1}{5}}.$$

The positive root also makes the helix rise as t increases, satisfying the stated orientation.

Question 10

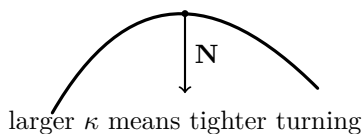
Let $\mathbf{r}(s)$ be a twice-differentiable curve parametrized by arc length, so $\|\mathbf{r}'(s)\| = 1$.

Tasks

1. Prove that $\mathbf{r}''(s)$ is perpendicular to $\mathbf{r}'(s)$.
2. Prove that $\kappa(s) = \|\mathbf{r}''(s)\|$.
3. Explain what $\mathbf{r}''(s) = \mathbf{0}$ means locally.

Question 10 – Solution

Strategy. Differentiate the unit-speed invariant, then compare the definition $\kappa = \|d\mathbf{T}/ds\|$ with $\mathbf{T} = \mathbf{r}'$.



Step 1: Orthogonality Unit speed means

$$\mathbf{r}'(s) \cdot \mathbf{r}'(s) = 1.$$

Differentiate both sides:

$$2\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0.$$

Therefore

$$\boxed{\mathbf{r}'(s) \cdot \mathbf{r}''(s) = 0}.$$

Thus the change of the unit tangent has no tangential component.

Step 2: Curvature formula Because the parameter is arc length,

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s).$$

Consequently

$$\kappa(s) = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{r}''(s)\|,$$

so

$$\boxed{\kappa(s) = \|\mathbf{r}''(s)\|}.$$

Step 3: Zero second derivative If $\mathbf{r}'' = \mathbf{0}$ throughout a neighborhood, then $\mathbf{r}'$ is a constant unit vector there. Integrating gives $\mathbf{r}(s) = \mathbf{p} + s\mathbf{u}$ locally, a straight line traversed at unit speed. At a single point, $\mathbf{r}''(s_0) = \mathbf{0}$ means zero curvature at that point, but does not by itself force an entire neighboring segment to be straight.