

Question 1

A point $P = (x, y, z)$ is known only through its octant and its distances to the coordinate axes.

Given

$$x < 0, \quad y > 0, \quad z < 0,$$

$$d(P, x\text{-axis}) = \sqrt{41},$$

$$d(P, y\text{-axis}) = \sqrt{34},$$

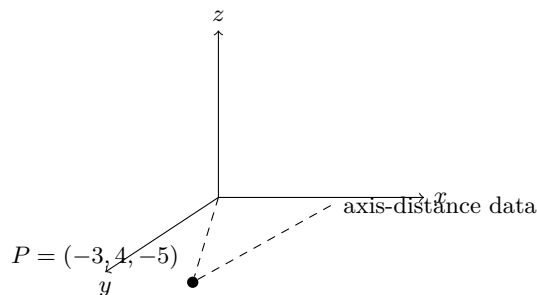
$$d(P, z\text{-axis}) = 5.$$

Tasks

1. Determine the coordinates of P .
2. Verify your result using all three axis distances.
3. Explain why only two of the three distance measurements would not determine P uniquely.

Question 1 – Solution

Strategy Squared distance to an axis is the sum of the squares of the other two coordinates. Solve first for x^2, y^2, z^2 , then use the octant to determine signs.



Equations The data give

$$y^2 + z^2 = 41, \quad x^2 + z^2 = 34, \quad x^2 + y^2 = 25.$$

Adding the last two equations and subtracting the first yields $2x^2 = 18$, so $x^2 = 9$. Similarly,

$$2y^2 = 41 + 25 - 34 = 32, \quad 2z^2 = 41 + 34 - 25 = 50.$$

Thus $|x| = 3$, $|y| = 4$, and $|z| = 5$. The stated octant fixes the signs:

$$\boxed{P = (-3, 4, -5)}.$$

Verification The resulting axis distances are

$$\sqrt{4^2 + (-5)^2} = \sqrt{41}, \quad \sqrt{(-3)^2 + (-5)^2} = \sqrt{34}, \quad \sqrt{(-3)^2 + 4^2} = 5.$$

With only two distances, one squared coordinate would remain free, so the third is essential.

Question 2

A sphere passes through four specified points in three-dimensional space.

Given

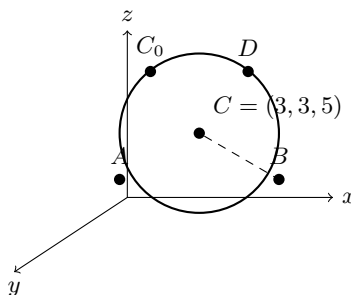
$$\begin{aligned}A &= (1, 0, 2), & B &= (5, 0, 2), \\C &= (1, 6, 2), & D &= (1, 0, 8).\end{aligned}$$

Tasks

1. Find the center and radius of the sphere.
2. Write its Cartesian equation.
3. Explain why the four given points determine a unique sphere.
4. Verify that all four points satisfy your equation.

Question 2 – Solution

Strategy A sphere's center is equidistant from all four points. Perpendicular-bisector conditions can be obtained by equating squared distances.



Center Let the center be $K = (h, k, \ell)$. From $d(A, K)^2 = d(B, K)^2$,

$$(h - 1)^2 + k^2 + (\ell - 2)^2 = (h - 5)^2 + k^2 + (\ell - 2)^2,$$

so $h = 3$. Comparing A with C gives $k = 3$, and comparing A with D gives $\ell = 5$. Hence the center is $(3, 3, 5)$ and

$$r^2 = (3 - 1)^2 + (3 - 0)^2 + (5 - 2)^2 = 22.$$

Therefore

$$\boxed{(x - 3)^2 + (y - 3)^2 + (z - 5)^2 = 22}.$$

Uniqueness and verification The three difference vectors from A point independently in the x -, y -, and z -directions, so their perpendicular-bisector planes meet in exactly one point. Substitution shows that each given point produces squared distance 22.

Question 3

Consider the fixed points

$$A = (-2, 0, 0), \quad B = (4, 0, 0),$$

and let $P = (x, y, z)$ satisfy the distance condition

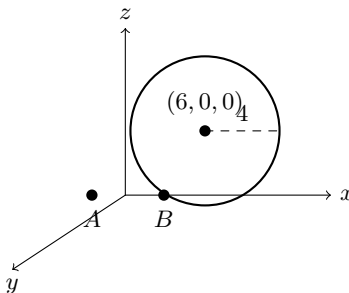
$$d(P, A) = 2d(P, B).$$

Tasks

1. Derive a Cartesian equation for the locus of all possible points P .
2. Identify the surface by stating its center and radius.
3. Determine which of A and B lies inside the surface.
4. Explain geometrically what the factor 2 requires of points on the locus.

Question 3 – Solution

Strategy Square the distance relation, expand, and complete the square. The resulting surface is an Apollonius sphere.



Derivation The condition is

$$(x + 2)^2 + y^2 + z^2 = 4((x - 4)^2 + y^2 + z^2).$$

After collecting terms and dividing by 3,

$$x^2 - 12x + y^2 + z^2 + 20 = 0.$$

Completing the square gives

$$\boxed{(x - 6)^2 + y^2 + z^2 = 16}.$$

Thus the locus is the sphere centered at $(6, 0, 0)$ with radius 4.

Interpretation Point B is 2 units from the center and lies inside; A is 8 units from the center and lies outside. A point must remain closer to B because its distance to A is required to be twice as large.

Verification At the axial endpoints $(2, 0, 0)$ and $(10, 0, 0)$, the distance pairs are $(4, 2)$ and $(12, 6)$, respectively, confirming the ratio.

Question 4

Let S be the set of points whose distance to the xy -plane is no greater than their distance to the z -axis:

$$S = \{P : d(P, xy\text{-plane}) \leq d(P, z\text{-axis})\}.$$

Tasks

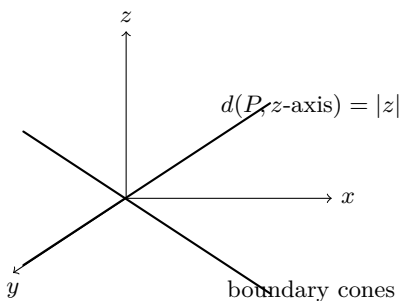
1. Derive a Cartesian inequality describing S .
2. Describe the boundary surface of S .
3. Determine whether each test point belongs to S :

$$P_1 = (1, 2, 3), \quad P_2 = (2, -2, 1).$$

4. Describe the reflectional and rotational symmetries of S .

Question 4 – Solution

Strategy Translate each geometric distance into coordinates: distance to the xy -plane is $|z|$, while distance to the z -axis is the radial distance $\sqrt{x^2 + y^2}$.



Inequality and boundary We require

$$|z| \leq \sqrt{x^2 + y^2} \iff \boxed{z^2 \leq x^2 + y^2}.$$

The boundary $z^2 = x^2 + y^2$ is a double cone with vertex at the origin. The set S is the region on or outside that cone in the radial direction.

Membership For $(1, 2, 3)$, $z^2 = 9 > 1^2 + 2^2 = 5$, so it is not in S . For $(2, -2, 1)$, $z^2 = 1 \leq 8$, so it is in S .

Symmetry and verification The inequality uses only squares, so changing any coordinate's sign preserves membership; rotations about the z -axis also preserve $x^2 + y^2$. On the xy -plane, every point satisfies it, while nonzero points on the z -axis do not, matching the distance interpretation.

Question 5

A rectangular laboratory is aligned with the coordinate axes and has opposite corners

$$O = (0, 0, 0), \quad V = (8, 6, 4).$$

A wireless transmitter will be installed at a point P inside the laboratory. Define

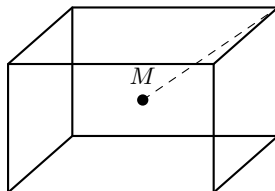
$$R(P) = \max\{d(P, C) : C \text{ is one of the eight corners}\}.$$

Design objective

1. Find the location of P that minimizes $R(P)$.
2. Find the minimum possible value of $R(P)$.
3. Prove that no other transmitter location can produce a smaller greatest distance.

Question 5 – Solution

Strategy Opposite corners force a lower bound: for any transmitter location, at least one of its distances to an opposite pair must be at least half the pair's separation. The box center attains that bound for every pair.



Lower bound The space diagonal has length

$$d(O, V) = \sqrt{8^2 + 6^2 + 4^2} = \sqrt{116} = 2\sqrt{29}.$$

For any location P , the triangle inequality gives

$$d(O, V) \leq d(O, P) + d(P, V).$$

Therefore at least one of $d(O, P)$ and $d(P, V)$ is at least $\sqrt{29}$.

Attaining the bound The midpoint of every space diagonal is

$$M = (4, 3, 2).$$

Each corner differs from M by coordinates $\pm 4, \pm 3, \pm 2$, so every corner is exactly

$$\sqrt{4^2 + 3^2 + 2^2} = \sqrt{29}$$

units away. Hence the optimal location is $\boxed{(4, 3, 2)}$, and the minimized greatest distance is $\boxed{\sqrt{29}}$.

Verification The lower bound and attained value agree, which proves global optimality rather than merely checking a plausible point.

Question 6

Two points determine a directed line segment:

$$A = (-4, 1, 7), \quad B = (6, -3, -1).$$

Tasks

1. Find the point P that divides $\overline{AB}$ internally in the ratio

$$AP : PB = 2 : 3.$$

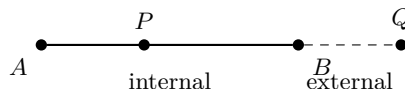
2. Find the point Q that divides the same line externally in the ratio

$$AQ : QB = 2 : 3.$$

3. Verify both ratios using three-dimensional distances, and distinguish clearly between internal and external division.

Question 6 – Solution

Strategy Parameterize the line by $X(t) = A + t(B - A)$. Internal division has $0 < t < 1$; external division requires t outside that interval.



Internal point Since $B - A = (10, -4, -8)$ and $AP/AB = 2/(2 + 3) = 2/5$,

$$P = A + \frac{2}{5}(B - A) = (-4, 1, 7) + (4, -8/5, -16/5) = \boxed{(0, -3/5, 19/5)}.$$

External point For $Q = A + t(B - A)$ beyond A , take $t < 0$. The ratio is

$$\frac{AQ}{QB} = \frac{-t}{1-t} = \frac{2}{3},$$

which gives $t = -2$. Thus

$$Q = A - 2(B - A) = \boxed{(-24, 9, 23)}.$$

Verification Because $P - A = (2/5)(B - A)$ and $B - P = (3/5)(B - A)$, their lengths have ratio $2 : 3$. Also $Q - A = -2(B - A)$ and $B - Q = 3(B - A)$, again giving $AQ : QB = 2 : 3$.

Question 7

Find every sphere satisfying all of the following conditions:

1. The sphere lies entirely in the octant

$$x \geq 0, \quad y \leq 0, \quad z \geq 0.$$

2. It is tangent to each of the three coordinate planes.

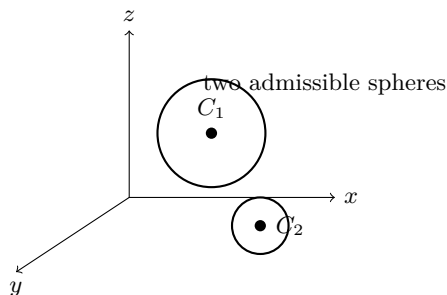
3. It passes through the point

$$P = (1, -2, 3).$$

For each sphere, state its center, radius, and Cartesian equation. Then give a geometric explanation for why two different spheres satisfy the same conditions.

Question 7 – Solution

Strategy Tangency to all three coordinate planes makes the center's coordinate magnitudes equal to the radius. The specified octant fixes their signs.



Center condition If the radius is $r > 0$, then the center must be

$$C = (r, -r, r).$$

Since P lies on the sphere,

$$(1 - r)^2 + (-2 + r)^2 + (3 - r)^2 = r^2.$$

Expanding gives $2r^2 - 12r + 14 = 0$, or

$$r^2 - 6r + 7 = 0.$$

Hence $r = 3 \pm \sqrt{2}$, both positive.

Results The two spheres are

$$(x - r)^2 + (y + r)^2 + (z - r)^2 = r^2, \quad r = 3 \pm \sqrt{2}.$$

Why two? Along the line of possible centers $(r, -r, r)$, the distance to P first decreases and then increases, so the equation “distance to center equals r ” can be met at two scales. Direct substitution verifies both roots, and each sphere has extreme coordinates touching—but not crossing—the three coordinate planes.

Question 8

A transformation T is performed in this order:

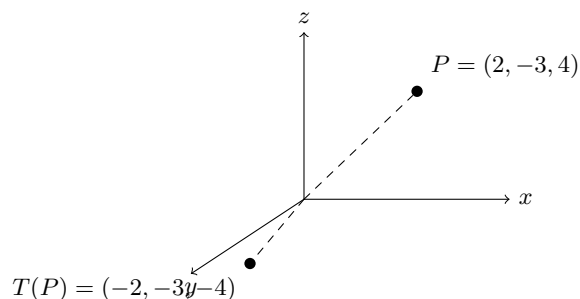
1. reflect across the yz -plane;
2. then reflect the result across the xy -plane.

Tasks

1. Find a coordinate formula for $T(x, y, z)$.
2. Compute $T(2, -3, 4)$.
3. Identify the single rigid motion equivalent to this composition.
4. Prove directly from the three-dimensional distance formula that T preserves the distance between every pair of points.

Question 8 – Solution

Strategy Track which coordinate changes sign under each reflection. Then compare squared distances before and after the transformation.



Coordinate rule Reflection across the yz -plane changes x to $-x$; reflection across the xy -plane changes z to $-z$. Therefore

$$\boxed{T(x, y, z) = (-x, y, -z)}, \quad T(2, -3, 4) = \boxed{(-2, -3, -4)}.$$

Geometric interpretation The y -coordinate is fixed while both perpendicular coordinates reverse. This is a rotation through 180° about the y -axis.

Distance preservation For $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$,

$$d(TP, TQ)^2 = (-x_1 + x_2)^2 + (y_1 - y_2)^2 + (-z_1 + z_2)^2 = d(P, Q)^2.$$

Both distances are nonnegative, so they are equal.

Verification Applying T twice restores every coordinate, as expected for a half-turn.

Question 9

A rectangular box has edges parallel to the coordinate axes and opposite vertices

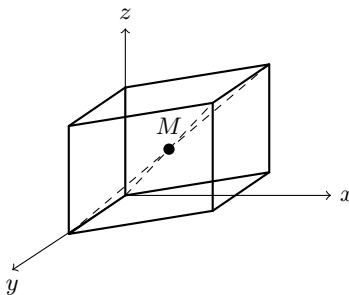
$$A = (1, -2, 3), \quad G = (7, 4, 13).$$

Tasks

1. List all eight vertices systematically.
2. Show that the four space diagonals have a common midpoint, and find it.
3. Find the center and radius of the sphere passing through all eight vertices.
4. Verify that every vertex lies on this sphere without substituting the vertices one at a time.

Question 9 – Solution

Strategy Each vertex independently chooses one of the two available values for each coordinate. Opposite vertices choose complementary values.



Vertices The coordinate choices are $x \in \{1, 7\}$, $y \in \{-2, 4\}$, $z \in \{3, 13\}$. Thus the vertices are

$$(1, -2, 3), (1, -2, 13), (1, 4, 3), (1, 4, 13),$$

$$(7, -2, 3), (7, -2, 13), (7, 4, 3), (7, 4, 13).$$

Common midpoint Every opposite pair averages the low and high value in each coordinate, giving

$$M = \left(\frac{1+7}{2}, \frac{-2+4}{2}, \frac{3+13}{2} \right) = \boxed{(4, 1, 8)}.$$

Hence all four space diagonals bisect one another at M .

Circumsphere From M to any vertex, the coordinate changes have magnitudes 3, 3, 5. Therefore

$$\boxed{r = \sqrt{3^2 + 3^2 + 5^2} = \sqrt{43}}.$$

Verification Every vertex has one of the sign patterns $(\pm 3, \pm 3, \pm 5)$ relative to M , so all eight lie on the sphere.

Question 10

A proposed sphere is required to satisfy all three conditions below:

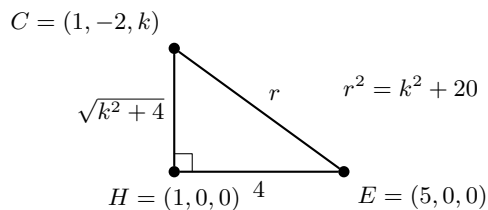
1. It intersects the x -axis at $A = (-3, 0, 0)$ and $B = (5, 0, 0)$.
2. Its center lies in the plane $y = -2$.
3. It is tangent to the xy -plane.

Consistency test

Determine whether such a sphere exists. If it does, find every possible center, radius, and equation. If it does not, prove precisely which requirements are incompatible.

Question 10 – Solution

Strategy The center lies on the perpendicular-bisector plane of the two axial intersection points. Tangency to the xy -plane relates the radius to the center's height.



Center coordinates The midpoint of the two intersections has x -coordinate 1, so write the center as

$$C = (1, -2, k).$$

The squared radius, using $(5, 0, 0)$, is

$$r^2 = (5 - 1)^2 + (0 + 2)^2 + k^2 = 20 + k^2.$$

Tangency to the xy -plane requires $r = |k|$, hence $r^2 = k^2$. Combining these equations would give $k^2 = 20 + k^2$, an impossibility.

Conclusion Therefore no sphere satisfies all the stated conditions.

Consistency review The two axial intersection points already force every possible center onto $x = 1$. Requiring $y = -2$ means the center has a nonzero horizontal offset from the x -axis. Consequently the radius needed to reach either axial point is strictly greater than $|k|$, while tangency to the xy -plane requires it to equal $|k|$. The contradiction is geometric, not an algebraic accident.