

Question 10

Let E be a bounded solid with piecewise smooth closed boundary S and outward unit normal $\mathbf{n}$.

Tasks

1. Use the Divergence Theorem with a suitable vector field to prove $\text{vol}(E) = \frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS$.
2. Apply the identity to a right circular cone of base radius R and height h , with vertex at the origin and base in the plane $z = h$.
3. Explain why the lateral surface contributes zero.

Question 10 – Solution

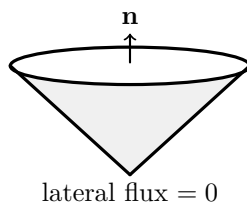
Strategy. Choose a field with divergence 1, then exploit the fact that the cone's radial position vector is tangent to its lateral generators.

Step 1: Derive the identity Let

$$\mathbf{G} = \frac{1}{3} \langle x, y, z \rangle.$$

Then $\nabla \cdot \mathbf{G} = 1$, so

$$\text{vol}(E) = \iiint_E 1 \, dV = \iint_S \mathbf{G} \cdot \mathbf{n} \, dS = \boxed{\frac{1}{3} \iint_S \langle x, y, z \rangle \cdot \mathbf{n} \, dS}.$$



Step 2: Lateral surface Every ray from the origin along the cone is tangent to the lateral surface. Hence $\langle x, y, z \rangle$ is tangent there and

$$\langle x, y, z \rangle \cdot \mathbf{n} = 0.$$

Step 3: Base On the base, $z = h$ and $\mathbf{n} = \mathbf{k}$, so the dot product equals h . Therefore

$$\boxed{\text{vol}(E) = \frac{1}{3} \iint_{\text{base}} h \, dA = \frac{1}{3} h (\pi R^2) = \frac{1}{3} \pi R^2 h}.$$

Verification The formula has the expected “one-third base times height” form and correct cubic units.